

Finding λ -translating solitons in the Lorentz-Minkowski Plane

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Mean Curvature Flow

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As an alternative to solving the PDE directly, we may consider special classes of surfaces that are much easier to find, such as *translating solitons*.

Example (Translating solitons in $\mathbb{R}^2$, [1])

- (1) Trivial example: a straight line.
- (2) The *grim reaper curve* $y = -\log |\cos x|$ is a translating soliton.

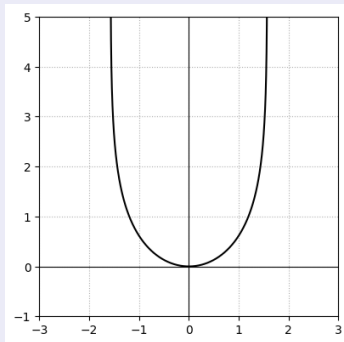


Figure: The grim reaper curve

The grim reaper curve translating parallel to $v = (0, 1)$ is a MCF. Thus, its shape is invariant under the MCF.

λ -translating Solitons

In this research, we look into a more general, perturbed version of the translating solitons:

Definition

A λ -*translating soliton* with respect to some unit vector v in $\mathbb{R}^n$ is a smooth, oriented hypersurface in $\mathbb{R}^n$ whose mean curvature vector $\mathbf{H}$ satisfies

$$\mathbf{H} = v^\perp + \lambda \mathbf{N},$$

where λ is some real number and $\mathbf{N}$ is the normal vector to the hypersurface.

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In [6], λ -translating solitons in $\mathbb{R}^2$ and $\mathbb{R}^3$ were found and classified. We are interested in finding λ -translating solitons in $\mathbb{L}^2$, the Lorentz-Minkowski plane.

Finding λ -translating Solitons in $\mathbb{L}^2$

The *Lorentz-Minkowski plane* $\mathbb{L}^2$ is the metric space $(\mathbb{R}^2, \langle \cdot, \cdot \rangle)$ where

$$\langle (x_1, y_1), (x_2, y_2) \rangle = x_1 x_2 - y_1 y_2.$$

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Applying this metric, we can turn the condition for λ -translating solitons with respect to $v \in \mathbb{L}^2$ into simple ODEs:

$$\varphi'(s) = a \sinh \varphi(s) - b \cosh \varphi(s) + \lambda$$

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We are able to find all solutions to the above equations and deduce the λ -translating solitons from the function φ .

The following are numerical plots of λ -translating solitons found in this research.

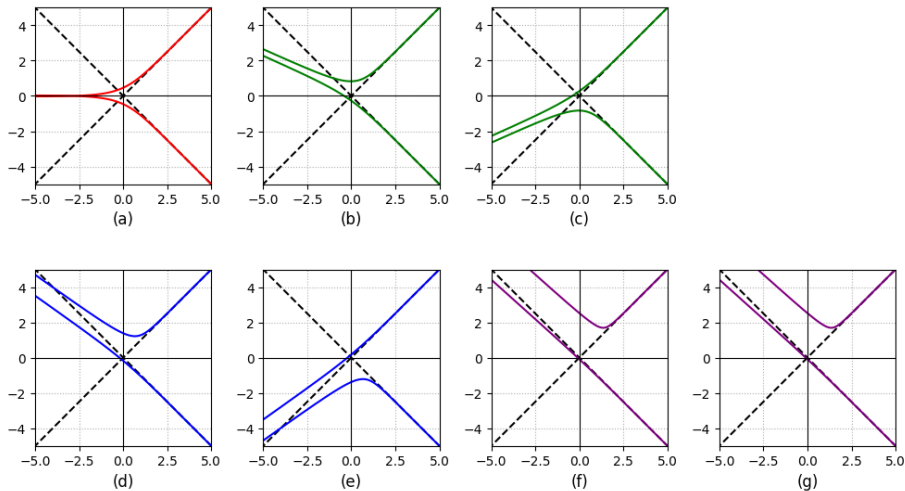


Figure: Plots of spacelike λ -translating solitons with respect to $(1,0)$: (a) $\lambda = 0$, (b) $\lambda = 1/2$, (c) $\lambda = -1/2$, (d) $\lambda = 1$, (e) $\lambda = -1$, (f) $\lambda = 2$, (g) $\lambda = -2$

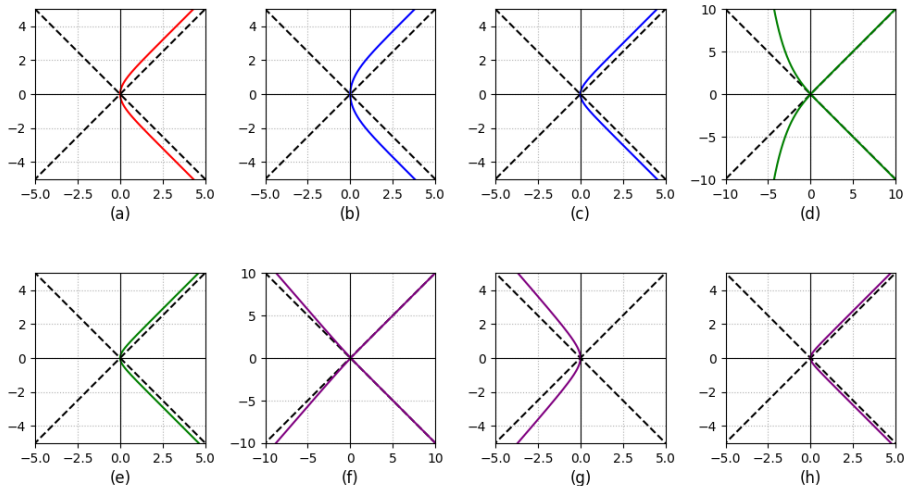


Figure: Plots of timelike λ -translating solitons with respect to $(1,0)$: (a) $\lambda = 0$, (b) $\lambda = 1/2$, (c) $\lambda = -1/2$, (d) $\lambda = 1$, (e) $\lambda = -1$, (f), (g) $\lambda = 2$, (h) $\lambda = -2$

Significance

Using the results of this research:

- mathematicians will be able to find λ -translating solitons in higher dimensions of the Lorentz-Minkowski space,
- gain further understanding of the mean curvature flow in various spaces.

References

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- [6] R. López, *Invariant surfaces in Euclidean space with a log-linear density*, Advances in Mathematics **339** (2018), 285-309.

Any questions?

Thank you!