

Linear Algebra

1st Edition

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Preface

Linear algebra is one of the first mathematics course that undergraduates encounter within their first two years. Unlike calculus, which is more concrete and intuitive, linear algebra is rather abstract. So an intensive course on linear algebra may turn out to be difficult to many students who are not yet familiar with college-level mathematics.

As a result, many elementary-level linear algebra courses choose to focus on numerical calculations rather than rigorous mathematical arguments. For example, most elementary linear algebra textbooks introduce determinants merely as a "special" function that outputs a real number for any square matrices. They do not care about what the value means; they only care about how to calculate determinants. Such approach towards linear algebra fails to deliver the significance of the subject.

Hence, I persisted with including rigorous mathematical contents although it may seem abstract and difficult to most readers. But I tried to make these contents merge naturally with elementary concepts by explaining the motivation of each definition and theorem, so that the reader can appreciate the necessity of each topic. Through this approach, the reader will gain a concrete understanding on linear algebra. It will motivate them to link distinct ideas and accept linear algebra as a solid subject. And this should be the primary aim of other linear algebra courses too.

A huge part of this book has been motivated by a 2021 fall semester course on Linear Algebra at Korean Minjok Leadership Academy. I express my deepest appreciation to all teachers and peers who helped me whilst writing this book.

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Introduction

What is linear algebra? First, the term 'linear' refers to something that resembles a linear structure such as a line or a flat plane. You may recall that the function $f(x) = ax$, whose graph is a line, is called a linear function. A linear function holds two main algebraic properties:

$$f(x_1 + x_2) = f(x_1) + f(x_2) \quad \text{and} \quad f(cx_1) = cf(x_1)$$

for all real numbers c , x_1 , and x_2 . These two properties are the common characteristics of all linear structures. Hence, the algebraic definition of 'linear' is consisted of these properties.

In the meantime, 'algebra' refers to a branch of mathematics that studies algebraic structures: nonempty sets of mathematical objects where operations are defined and satisfy a certain set of algebraic rules.

Hence, linear algebra, the combination of these two terms, refers to the study of algebraic structures that are linear. More specifically, linear algebra is the study of finite-dimensional vector spaces. So we begin the book with a discussion on what vectors and vector spaces are. The first two chapters are devoted to this:

- In Chapter 1, we look at a special type of vectors: vectors in $\mathbb{R}^n$. These vectors are visual and intuitive; so, they are an appropriate starting point for our study. We cover the algebraic and geometric representations of vectors, its operations, especially inner products, and the significance of vectors in analytic geometry. A brief introduction to orthogonal decomposition, vector representation of lines and planes in $\mathbb{R}^n$, and cross products is given at the end of the chapter.
- Chapter 2 deals with the most general notion of vectors. Based on the algebraic properties of vectors in $\mathbb{R}^n$, we construct a solid definition for vector spaces, a nonempty set of vectors where addition and scalar multiplication are defined to satisfy a list of axioms. Then, we move to the basic theories of vector spaces: span, linear independence, basis, and dimension. We shall examine what each concept means, and how they describe the properties of vector spaces.

The next two chapters introduce two core topics of linear algebra: linear systems and linear maps. A linear system is a system of linear equations, and a linear

map is a function between vector spaces that holds linearity. And when describing these two, we use a mathematical tool called matrices. Hence:

- Chapter 3 starts with what matrices are. We define its operations, and dive deeper into matrix algebra. Then we move to the topic of linear systems. We look at how to solve linear systems via Gauss-Jordan elimination and when linear systems have either no solution, exactly one solution, or infinitely many solutions. The core of this chapter is the relation between linear systems and matrix algebra. The invertibility of a matrix determines how the solution set of the represented linear system looks like. We end with a brief discussion on the fundamental subspaces of matrices.
- Chapter 4 concerns linear maps and operations between vector spaces. We see how linear maps can be represented by matrices, and why matrix operations allow us to find composite maps and inverse maps. With inverse maps follows the concept of isomorphism and conditions for a linear map to be invertible. Finally, we introduce a fundamental theorem of linear maps concerning similarity. We will check that different matrices representing the same linear map share certain properties.

With a concrete understanding of linear systems and linear maps, we focus on square matrices / linear operators for the final two chapters.

- Chapter 5 introduces the determinant - a function that assigns real numbers to all square matrices. We discuss the meaning of the determinant (in terms of linear maps), and derive the explicit formula of the determinant based on three axioms. Using the formula, we prove several properties of the determinant and discuss the relation between determinants and the invertibility of a square matrix. Finally, we observe the link between determinants and linear systems; Cramer's rule explains this relationship.
- Chapter 6 covers the diagonalization problem: when is a given square matrix similar to a diagonal matrix? To solve this problem, we first introduce eigenvalues and eigenvectors, which are scalars and vectors with special properties. We shall practice methods to find the eigenvalues and eigenvectors of a given square matrix. Finally, we formally deal with when a square matrix is diagonalizable and how the process of diagonalization works.

To this end, we cover the major topics of linear algebra. But regard that this book is only an introductory course. There are more novelties that linear algebra is capable of providing.

For example, another core topic of linear algebra is orthogonality and inner product spaces. The inner product is defined in the most general sense, and the concept of orthogonality is extended to general vector spaces. Linear operators on inner product spaces, such as the self-adjoint operator and the normal operator are also investigated under this topic.

Although this book restrictively deals with vector spaces over the real number set $\mathbb{R}$ (real vector spaces), vector spaces can actually be defined over any field, including the complex number set $\mathbb{C}$. Matrices with complex entries become involved in this context, and thus, hermitian, unitary, and normal matrices arise. This leads us to a completely new branch of investigation.

For computational purposes, some mathematicians deeply engage with matrix analysis, which focuses on matrix algebra and the computation of matrix operations. In this branch, decomposition methods such as the QR-decomposition, LU-decomposition, and the singular value decomposition are introduced, and researchers further investigate new methods to compute certain matrix operations, such as the Gauss-Jordan elimination, with smaller time complexity.

Linear algebra is also useful in other branches of mathematics such as differential equations. It turns out that the solution set of linear differential equations are vector spaces, on which we can apply linear algebra concepts freely. Also, matrices can be used to solve a system of linear differential equations, and from here we obtain a motivation for the Jordan canonical form.

This is just a small part of what linear algebra is capable of. Whichever branch of mathematics you choose to study in the future, your knowledge on linear algebra will always come in useful. Hence, allow this book to guide you through linear algebra.

1 | Vectors in $\mathbb{R}^n$

As stated in the Introduction, linear algebra is the study of finite-dimensional vector spaces. So it is necessary that we begin with a firm understanding of what vectors are.

What is a vector? If you have already studied physics before, you may be familiar with the statement: "a vector is a physical quantity with both magnitude and direction". And you would have probably drawn an arrow into an empty space to denote a given vector.

This was how both mathematicians and physicians accepted vectors in the beginning. Simply, just an arrow pointing to some direction in the space. But as time passed, people found that instead of using arrows, we can use a list of numbers to fully express vectors. This was based on the concept of coordinates.

The usage of coordinates made it easier to calculate vector operations and apply vectors to geometry in $\mathbb{R}^2$ and $\mathbb{R}^3$ space. It even allowed mathematicians to extend vectors to the $\mathbb{R}^n$ space, which is theoretically a space consisted of n perpendicular axes, and is not visualizable.

Hence, in this chapter, we start off with how people came up with vectors. We will see that vectors can be expressed as both arrows and coordinates. Then we will observe how vector operations are defined, and how mathematicians defined vectors in $\mathbb{R}^n$ space.

1.1 Vectors as "Arrows"

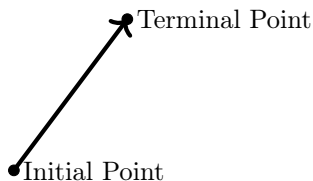


Figure 1.1: Arrow Representing a Vector

In physics, a **vector** is a physical quantity with direction and magnitude. Examples are force, velocity, and momentum.

We express vectors using arrows. Given two points, the initial point and the terminal point, the arrow must be connecting the two points as a straight line, with the head of the arrow pointing towards the terminal point.

The direction of the arrow head represents the vector's direction, and the length of the arrow represents the vector's magnitude.

For simplicity, we also use boldface letters like $\mathbf{u}, \mathbf{v}, \mathbf{w}, \dots$ to represent vectors. A vector with magnitude zero is called a **zero vector**, and denoted as $\mathbf{0}$. A zero vector is considered not to have a specific direction, for its initial point and terminal point coincide, leaving its direction undetermined.

Vector Operations

Given two arrows of the same direction and length, we say that the arrows represent the same vector, regardless of the position of the arrow. This makes sense, because the position of the arrow doesn't affect the direction and length of the arrow. As long as those two components are equivalent, the vectors should also be claimed equivalent.

Now suppose we have two vectors $\mathbf{u}$ and $\mathbf{v}$. To find the **sum** of $\mathbf{u}$ and $\mathbf{v}$, denoted $\mathbf{u} + \mathbf{v}$, begin by translating $\mathbf{v}$ so that the initial point of $\mathbf{v}$ coincides with the terminal point of $\mathbf{u}$, as given in Figure 1.2.

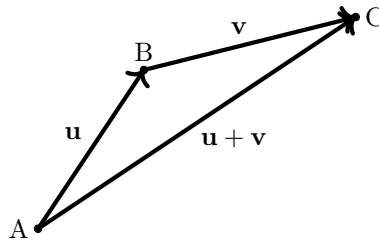


Figure 1.2: Vector Addition

Imagine each vector represents the movement of a particle. A particle at A will move to B due to $\mathbf{u}$, and then to C due to $\mathbf{v}$. Hence, $\mathbf{u}$ and $\mathbf{v}$ combined, the particle will move from A to C .

Following this intuition, we may define $\mathbf{u} + \mathbf{v}$ as the vector represented by an arrow connecting A , C and directing towards C . Hence we have what is called the **triangle rule** of vector addition.

Now let vector $\mathbf{u}$ be given. For some real number c , the **scalar multiple** of c and $\mathbf{u}$, denoted $c\mathbf{u}$, is a vector whose arrow is parallel to $\mathbf{u}$, but whose length is multiplied by $|c|$. If $c > 0$, the vector is in the same direction as $\mathbf{u}$, and if $c < 0$, the vector is in the opposite direction as $\mathbf{u}$. If $c = 0$, $c\mathbf{u}$ is always the zero vector $\mathbf{0}$. Figure 1.3 gives an illustration.

Notice that scalar multiples come from the addition of vectors. For example, $2\mathbf{u}$ represents twice of $\mathbf{u}$, in other words, the sum $\mathbf{u} + \mathbf{u}$. By the prior definition

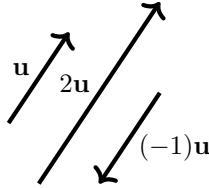


Figure 1.3: Scalar Multiplication

of vector addition, we can show that $\mathbf{u} + \mathbf{u}$ is a vector in the same direction as $\mathbf{u}$ with double its magnitude.

Then how do we interpret when $c < 0$? We need to first look at vector subtractions. Let vector $\mathbf{u} + \mathbf{v}$ in Figure 1.2 be denoted as $\mathbf{w}$. Since $\mathbf{w}$ is the sum of $\mathbf{u}$ and $\mathbf{v}$, we can say that $\mathbf{v}$ is a vector made by subtracting $\mathbf{u}$ from $\mathbf{w}$. In terms of equations, we can write

$$\mathbf{v} = \mathbf{w} - \mathbf{u}.$$

Notice that if $-\mathbf{u}$, the **negative** of $\mathbf{u}$, is a vector of same magnitude as $\mathbf{u}$ but with direction opposite to $\mathbf{u}$, then

$$\mathbf{v} = \mathbf{w} + (-\mathbf{u}).$$

Hence, vector subtractions can be expressed as vector additions.

Consider the scalar multiple $(-1)\mathbf{u}$. From the idea that $2\mathbf{u} = \mathbf{u} + \mathbf{u}$, we can extend that $(-1)\mathbf{u} = -\mathbf{u}$, since multiplying -1 tends to change the sign of the other operand, while preserving its absolute value. Hence, if $c < 0$, the scalar multiple $c\mathbf{u}$ always goes in the opposite direction of $\mathbf{u}$, for it must be in the same direction as $-\mathbf{u}$.

So far we have studied vectors in its geometric perspective. Accordingly, we defined vector operations in the sense that there are no logical contradictions. Now we look at vectors in the algebraic perspective, which will be the more useful setting in linear algebra.

1.2 Vectors as "Coordinates"

We are already familiar with how the real coordinate system works. The coordinate plane, or the two-dimensional coordinate space, has two axes; the x -axis and y -axis, which are perpendicular to each other. The three-dimensional coordinate space has three axes; the x -axis, y -axis, and z -axis, which are also perpendicular to each other for any distinct pairs.

The coordinate system defines coordinates, a list of real numbers, for every point above its coordinate space. Each real number in the list comes from a single axis. Hence, the coordinate of a point in the coordinate plane is a list of two real numbers, and the coordinate of a point in the three-dimensional coordinate space is a list of three real numbers.

Now we can generalize that a n -dimensional coordinate space has n axes that

are perpendicular to each other, and each point above the coordinate space has coordinates consisted of n real numbers.

Definition 1.1. *Given a positive integer n , an **ordered n -tuple** is an array of n real numbers $(x_1, x_2, \dots, x_n)$. The set of all ordered n -tuples is denoted as*

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) : x_1, x_2, \dots, x_n \in \mathbb{R}\}.$$

Remark. *In the above definition, $\mathbb{R}^n$ can also be seen as the set of coordinates of all points in the n -dimensional coordinate space.*

Due to the least upper bound property of real numbers, the set of real numbers fills up a whole number line. Hence, a coordinate axis is entirely made up of real numbers, and the coordinate of any point on the coordinate space is consisted of real numbers.

A single ordered n -tuple corresponds to a single point above the n -dimensional coordinate space. So ordered n -tuples provide the perfect way to express points algebraically.

These ordered n -tuples can also be used to express vectors algebraically. For example, suppose we have a vector $\mathbf{u} = (a, b)$ with its initial point at the origin $(0, 0)$.

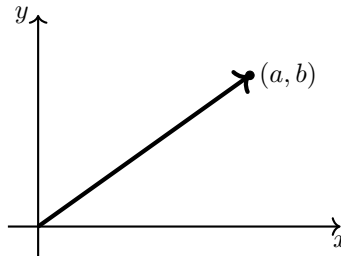


Figure 1.4: Vector on Coordinate Plane

If we unite the initial point of all vectors to $(0, 0)$, then a vector's direction and magnitude is only dependent on the terminal point, which becomes (a, b) for vector $\mathbf{u}$. Hence, all we need is the coordinate of the terminal point to express the vector. So we can write $\mathbf{u} = (a, b)$.

Even if a given vector's initial point is not on the origin, all we have to do is translate it, preserving its direction and magnitude, so that its initial point comes to the origin. Hence, we can express all vectors on the coordinate plane using ordered 2-tuples. This works same for higher dimensions, like the three-dimensional coordinate space, too.

This algebraic representation method of vectors has two advantages. First, we can generalize the notion of vectors to higher, non-visible dimensions like $\mathbb{R}^4$, which can not be geometrically interpreted. Second, it turns out that vector operations become much simple, as we will see below.

Vector Operations

We have seen that two vectors are equivalent when their directions and magnitudes are the same. In terms of coordinates, this means that their terminal points coincide when their initial points are on the origin. Hence, the ordered n -tuple representing the vectors must be the same for the vectors to be equivalent.

Definition 1.2. Let $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ be vectors in $\mathbb{R}^n$. We say that $\mathbf{u}$ and $\mathbf{v}$ are **equal** and denote $\mathbf{u} = \mathbf{v}$ if

$$u_i = v_i \quad \text{for all } i = 1, 2, \dots, n.$$

Example 1.3. Let $\mathbf{u} = (-1, a, 3)$ and $\mathbf{v} = (b, 2, 3)$ be vectors in $\mathbb{R}^3$. Then the equivalent statement to $\mathbf{u} = \mathbf{v}$ is $a = 2$ and $b = -1$.

Now let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be vectors in $\mathbb{R}^2$. Following the triangle rule of vector addition, translate $\mathbf{v}$ so that its initial point is (u_1, u_2) . Then, its terminal point becomes $(u_1 + v_1, u_2 + v_2)$. Therefore, vector $\mathbf{u} + \mathbf{v}$ would be $(u_1 + v_1, u_2 + v_2)$.

Notice that vector addition is the same as adding each component of the coordinates. Generalizing this result to vectors in $\mathbb{R}^n$ gives the following.

Definition 1.4. If $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ are vectors in $\mathbb{R}^n$, then the **sum** $\mathbf{u} + \mathbf{v}$ is a vector in $\mathbb{R}^n$ defined as

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n).$$

This time, we consider the scalar multiples of vectors. Let $\mathbf{u} = (u_1, u_2)$ be a vector in $\mathbb{R}^2$ again. Then, scalar multiple $2\mathbf{u}$ would have to be $(2u_1, 2u_2)$ since $\mathbf{u}$ and $2\mathbf{u}$ are parallel and $2\mathbf{u}$ has double the magnitude of $\mathbf{u}$.

Scalar multiple $-\mathbf{u}$ would be $(-u_1, -u_2)$ since $-\mathbf{u}$ is in the opposite direction of $\mathbf{u}$ and has the same magnitude.

Combining these results, we get that some real number c gives that $c\mathbf{u} = (cu_1, cu_2)$. The following is a general result for all vectors in $\mathbb{R}^n$.

Definition 1.5. If $\mathbf{u} = (u_1, u_2, \dots, u_n)$ is a vector in $\mathbb{R}^n$ and c is a scalar, then the **scalar multiple** $c\mathbf{u}$ is a vector in $\mathbb{R}^n$ defined as

$$c\mathbf{u} = (cu_1, cu_2, \dots, cu_n).$$

Notice that the definitions above are motivated from the addition and scalar multiplication of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, which we discovered in Section 1.1. Operations between vectors in the general $\mathbb{R}^n$ space is purely algebraic and defined via coordinates.

Example 1.6. Let $\mathbf{u} = (-1, 2, 3)$ and $\mathbf{v} = (3, 0, 2)$. Then

$$\mathbf{u} + \mathbf{v} = (-1 + 3, 2 + 0, 3 + 2) = (2, 2, 5)$$

and

$$3\mathbf{u} = (3 \cdot (-1), 3 \cdot 2, 3 \cdot 3) = (-3, 6, 9).$$

Next, we look at some important algebraic properties of the addition and scalar multiplication of vectors.

Theorem 1.7. *If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^n$ and c, d are scalars, then:*

- (a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- (b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
- (c) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$.
- (d) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- (e) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
- (f) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
- (g) $c(d\mathbf{u}) = (cd)\mathbf{u}$.
- (h) $1\mathbf{u} = \mathbf{u}$.

Proof. By Definition 1.2, it suffices to show that the corresponding ordered n -tuples for both sides of the equations are equivalent. Use Definition 1.4, 1.5, and the algebraic properties of real numbers to show the rest. ■

The theories of linear algebra depend on these eight algebraic properties, for all theorems are proved with these properties. This is what makes addition and scalar multiplication the most fundamental operations of vectors.

We have completed our observations on vectors in the geometric and algebraic perspectives. The two perspectives suggested different methods for expressing vectors, but turned out to give the same results.

Now we will look at the geometric application of vectors, especially in $\mathbb{R}^2$ and $\mathbb{R}^3$.

1.3 Vectors and Analytic Geometry

Vectors allow us to freely transition between its geometric and algebraic representations. Hence, we can either solve algebraic problems geometrically, or solve geometric problems algebraically. This leads us to study how useful vectors are in analytic geometry.

We first look at another important vector operation, which opens road to the geometric applications of vectors.

Definition 1.8. *If $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ are vectors in $\mathbb{R}^n$, then the **(Euclidean) inner product** (or **dot product**) of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \cdot \mathbf{v}$, is defined as*

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \dots + u_nv_n.$$

Example 1.9. Let $\mathbf{u} = (-1, 0, 2, 3)$ and $\mathbf{v} = (1, 3, -2, 4)$. Then

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= (-1) \cdot 1 + 0 \cdot 3 + 2 \cdot (-2) + 3 \cdot 4 \\ &= (-1) + 0 + (-4) + 12 = 7.\end{aligned}$$

The following theorem gives the basic algebraic properties of inner products. They will be used often in proofs for later theorems related to inner products.

Theorem 1.10. If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vectors in $\mathbb{R}^n$ and k is a scalar, then:

- (a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
- (b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.
- (c) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$.

Proof. Let $\mathbf{u} = (u_1, u_2, \dots, u_n)$, $\mathbf{v} = (v_1, v_2, \dots, v_n)$, and $\mathbf{w} = (w_1, w_2, \dots, w_n)$.

(a) Then, using the commutativity of multiplication between real numbers,

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= u_1v_1 + u_2v_2 + \dots + u_nv_n \\ &= v_1u_1 + v_2u_2 + \dots + v_nu_n = \mathbf{v} \cdot \mathbf{u}.\end{aligned}$$

Hence, $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

(b) This time, using the distribution law, we have

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= (u_1, u_2, \dots, u_n) \cdot (v_1 + w_1, v_2 + w_2, \dots, v_n + w_n) \\ &= u_1(v_1 + w_1) + u_2(v_2 + w_2) + \dots + u_n(v_n + w_n) \\ &= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + \dots + u_nv_n + u_nw_n \\ &= (u_1v_1 + u_2v_2 + \dots + u_nv_n) + (u_1w_1 + u_2w_2 + \dots + u_nw_n) \\ &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}.\end{aligned}$$

(c) Finally, using the distribution law again,

$$\begin{aligned}k(\mathbf{u} \cdot \mathbf{v}) &= k(u_1v_1 + u_2v_2 + \dots + u_nv_n) \\ &= ku_1v_1 + ku_2v_2 + \dots + ku_nv_n \\ &= (ku_1)v_1 + (ku_2)v_2 + \dots + (ku_n)v_n \\ &= (ku_1, ku_2, \dots, ku_n) \cdot (v_1, v_2, \dots, v_n) \\ &= (k\mathbf{u}) \cdot \mathbf{v}.\end{aligned}$$

Similarly, we can prove that $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$. ■

Inner products turn out to have various applications in analytic geometry. We shall look at some important ones.

Vector Norm

Definition 1.11. If $\mathbf{u} = (u_1, u_2, \dots, u_n)$ is a vector in $\mathbb{R}^n$, then the **norm** of $\mathbf{u}$, denoted by $\|\mathbf{u}\|$, is defined as

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}. \quad (1.1)$$

Example 1.12. The norm of $\mathbf{u} = (-1, 0, 2, 3)$ is

$$\|\mathbf{u}\| = \sqrt{(-1)^2 + 0^2 + 2^2 + 3^2} = \sqrt{14}.$$

The norm of a vector represents the magnitude of the vector. (1.1) is derived by Pythagoras' theorem for cases $n = 2, 3$, and Definition 1.11 is a generalization of those cases to all n .

The following theorem gives the basic properties of vector norms.

Theorem 1.13. If $\mathbf{u}$ is a vector in $\mathbb{R}^n$ and k is a scalar, then:

- (a) $\|\mathbf{u}\| \geq 0$.
- (b) $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = \mathbf{0}$.
- (c) $\|k\mathbf{u}\| = |k| \cdot \|\mathbf{u}\|$.

Proof. (a) Let $\mathbf{u} = (u_1, u_2, \dots, u_n)$. Since u_i^2 is greater than 0 for all i between 1 and n ,

$$u_1^2 + u_2^2 + \dots + u_n^2 \geq 0,$$

which gives

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2} \geq 0.$$

(b) ($\Rightarrow$) Let $\mathbf{u} = (u_1, u_2, \dots, u_n)$. Since

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2} = 0,$$

we have

$$u_1^2 + u_2^2 + \dots + u_n^2 = 0,$$

which gives $u_1 = u_2 = \dots = u_n = 0$. Hence, $\mathbf{u} = \mathbf{0}$.

($\Leftarrow$) If $\mathbf{u} = \mathbf{0}$, then it is obvious that $\|\mathbf{u}\| = 0$.

(c) Let $\mathbf{u} = (u_1, u_2, \dots, u_n)$. Then, $k\mathbf{u} = (ku_1, ku_2, \dots, ku_n)$. Hence,

$$\begin{aligned} \|k\mathbf{u}\| &= \sqrt{(ku_1)^2 + (ku_2)^2 + \dots + (ku_n)^2} \\ &= \sqrt{k^2(u_1^2 + u_2^2 + \dots + u_n^2)} \\ &= |k| \cdot \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}, \end{aligned}$$

which gives $\|k\mathbf{u}\| = |k| \cdot \|\mathbf{u}\|$. ■

Theorem 1.13 (a) is the same as saying that distance between two points is always nonnegative. Also, (b) tells us that the norm of a vector is zero only when the vector is a zero vector. Hence, all nonzero vectors have a positive norm. (c) turns out to be useful when calculating norms.

Theorem 1.14. *If $\mathbf{u}$ is a vector in $\mathbb{R}^n$, then*

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}. \quad (1.2)$$

Proof. Let $\mathbf{u} = (u_1, u_2, \dots, u_n)$. Then,

$$\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + \dots + u_n^2,$$

which gives $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$. Hence, we have (1.2) since $\|\mathbf{u}\|$ is always greater than 0. ■

Above theorem shows that inner products are used to calculate the vector norm. We often express vector norms as (1.2) when using the property of inner products turns out to be useful.

Definition 1.15. *A **unit vector** is a vector whose norm is equal to 1.*

Suppose $\mathbf{u}$ is a vector in $\mathbb{R}^n$. Then, a unit vector in the direction of $\mathbf{u}$ is denoted as $\mathbf{e}_\mathbf{u}$ and

$$\mathbf{e}_\mathbf{u} = \frac{\mathbf{u}}{\|\mathbf{u}\|}$$

since

$$\|\mathbf{e}_\mathbf{u}\| = \left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = \frac{1}{\|\mathbf{u}\|} \cdot \|\mathbf{u}\| = 1.$$

Imagine the unit circle $x^2 + y^2 = 1$ on the coordinate plane. Unit vectors are vectors with the initial point on the origin and the terminal point on the unit circle. Hence, a unit vector is unique for all directions.

We call

$$\mathbf{e}_1 = (1, 0, \dots, 0), \mathbf{e}_2 = (0, 1, \dots, 0), \dots, \mathbf{e}_n = (0, 0, \dots, 1)$$

the **standard unit vectors** in $\mathbb{R}^n$. Notice that $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ are unit vectors in the direction of each axis on the coordinate space. Given $\mathbf{u} = (u_1, u_2, \dots, u_n)$, a vector in $\mathbb{R}^n$, we can express it in terms of the standard unit vectors as

$$\mathbf{u} = (u_1, u_2, \dots, u_n) = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + \dots + u_n\mathbf{e}_n \quad (1.3)$$

where $u_1, u_2, \dots, u_n$ turn out to form the coordinate of $\mathbf{u}$. Later on, we will show that the coordinate of all vectors in $\mathbb{R}^n$ are uniquely determined by these standard unit vectors.

Theorem 1.16. *For standard unit vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$,*

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \quad \text{for all } i, j \in \{1, 2, \dots, n\}.$$

Proof. If $i = j$, then $\mathbf{e}_i \cdot \mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{e}_i = \|\mathbf{e}_i\|^2$. Since $\mathbf{e}_i$ is a unit vector, its norm is 1, and thus, $\mathbf{e}_i \cdot \mathbf{e}_j = 1$.

If $i \neq j$, then it can be directly showed that $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ by using the coordinates of $\mathbf{e}_i$ and $\mathbf{e}_j$. ■

Remark. δ_{ij} is a two-variable function of i and j called the **Kronecker delta**, defined as

$$\delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}.$$

Using Theorem 1.16 we get

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= (u_1, u_2, \dots, u_n) \cdot (v_1, v_2, \dots, v_n) \\ &= (u_1 \mathbf{e}_1 + \dots + u_n \mathbf{e}_n) \cdot (v_1 \mathbf{e}_1 + \dots + v_n \mathbf{e}_n) \\ &= \sum_{1 \leq i, j \leq n} u_i v_j (\mathbf{e}_i \cdot \mathbf{e}_j) = \sum_{1 \leq i, j \leq n} u_i v_j \delta_{ij} \\ &= \sum_{1 \leq i=j \leq n} u_i v_j \delta_{ij} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n, \end{aligned}$$

which matches with Definition 1.8.

The following theorem is a famous inequality applicable to all vectors in $\mathbb{R}^n$. It may be already familiar to you, but we shall review it to discuss what it implies.

Theorem 1.17 (Cauchy-Schwarz Inequality). *If $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ are vectors in $\mathbb{R}^n$, then*

$$\left(\sum_{i=1}^n u_i v_i \right)^2 \leq \left(\sum_{i=1}^n u_i^2 \right) \left(\sum_{i=1}^n v_i^2 \right).$$

Proof. Consider the polynomial

$$(u_1 x + v_1)^2 + (u_2 x + v_2)^2 + \dots + (u_n x + v_n)^2,$$

which can be expressed as

$$\left(\sum_{i=1}^n u_i^2 \right) x^2 + 2 \left(\sum_{i=1}^n u_i v_i \right) x + \left(\sum_{i=1}^n v_i^2 \right).$$

Since the polynomial is greater than or equal to 0 for all $x \in \mathbb{R}$, its discriminant D must be smaller than or equal to 0. Hence,

$$D = \left(\sum_{i=1}^n u_i v_i \right)^2 - \left(\sum_{i=1}^n u_i^2 \right) \left(\sum_{i=1}^n v_i^2 \right) \leq 0$$

and the theorem is proven. ■

The inequality in the above theorem can also be expressed as

$$(\mathbf{u} \cdot \mathbf{v})^2 \leq \|\mathbf{u}\|^2 \|\mathbf{v}\|^2,$$

which leads to

$$-\|\mathbf{u}\| \|\mathbf{v}\| \leq \mathbf{u} \cdot \mathbf{v} \leq \|\mathbf{u}\| \|\mathbf{v}\|$$

and

$$-1 \leq \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \leq 1. \quad (1.4)$$

We investigate further to see what the above equation gives.

Theorem 1.18. *Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then,*

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$

where θ is the angle between arrows representing $\mathbf{u}$ and $\mathbf{v}$ when their initial points coincide.

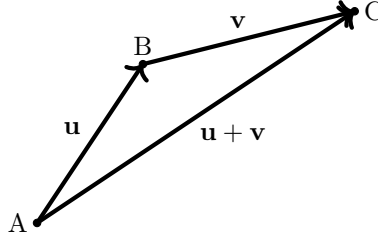


Figure 1.5: Proof to Theorem 1.18

Proof. Using Theorem 1.10, we get

$$\|\mathbf{v} - \mathbf{u}\|^2 = \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u}$$

which leads to

$$\|\mathbf{v} - \mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v}. \quad (1.5)$$

Now suppose we allocate arrows representing $\mathbf{u}, \mathbf{v}, \mathbf{v} - \mathbf{u}$ as given in Figure 1.5. Then, by the 2nd law of cosine, we get

$$\|\mathbf{v} - \mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta. \quad (1.6)$$

Finally, (1.5) and (1.6) leads to

$$\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta,$$

giving us the theorem. ■

By the above theorem, the inner product of two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ can be expressed in terms of their norms and the cosine of the angle between the vectors. Hence, for nonzero vectors $\mathbf{u}, \mathbf{v}$, the angle can be expressed as

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \quad (1.7)$$

or

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right). \quad (1.8)$$

Example 1.19. Consider $\mathbf{u} = (-1, 0, 2)$ and $\mathbf{v} = (4, 0, 3)$. Then, the angle between $\mathbf{u}$ and $\mathbf{v}$ is

$$\begin{aligned} \theta &= \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right) \\ &= \cos^{-1} \left(\frac{-1 \cdot 4 + 0 \cdot 0 + 2 \cdot 3}{\sqrt{(-1)^2 + 0^2 + 2^2} \cdot \sqrt{4^2 + 0^2 + 3^2}} \right) \\ &= \cos^{-1} \left(\frac{2}{5\sqrt{5}} \right). \end{aligned}$$

Notice that (1.4) gives justification to (1.7). Since the cosine function is always between -1 and 1 , (1.7) must also be between -1 and 1 , as indicated by (1.4).

Of course, (1.7) can be purely derived from Theorem 1.18. But if we were to generalize the concept of angles to $\mathbb{R}^n$, (1.4) would be useless since the equation only holds for vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. Meanwhile, (1.7) works for vectors in $\mathbb{R}^n$. And so we could use it as our justification to defining the angle between nonzero vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$ as (1.8).

Thus, we extend the notion of angles to $\mathbb{R}^n$, and Theorem 1.18 ends up working for all vectors in $\mathbb{R}^n$.

Definition 1.20. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, then the **angle** between $\mathbf{u}$ and $\mathbf{v}$ is defined as

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right).$$

Theorem 1.21. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Then,

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Distance Between Vectors

Given two vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$ with identical initial points, $\mathbf{u} - \mathbf{v}$ is a vector whose arrow connects the terminal points of both $\mathbf{u}$ and $\mathbf{v}$. Hence, the distance between the terminal points of $\mathbf{u}$ and $\mathbf{v}$ would become the norm of $\mathbf{u} - \mathbf{v}$. This leads us to the following definition.

Definition 1.22. If $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ are vectors in $\mathbb{R}^n$, then the **distance** between $\mathbf{u}$ and $\mathbf{v}$, denoted by $d(\mathbf{u}, \mathbf{v})$, is defined as

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \dots + (u_n - v_n)^2}.$$

Example 1.23. If $\mathbf{x} = (-1, 0, 2, 3)$ and $\mathbf{y} = (1, 3, -2, 4)$, then

$$\begin{aligned} d(\mathbf{x}, \mathbf{y}) &= \sqrt{(-1 - 1)^2 + (0 - 3)^2 + (2 - (-2))^2 + (3 - 4)^2} \\ &= \sqrt{4 + 9 + 16 + 1} = \sqrt{30}. \end{aligned}$$

Orthogonality

Recall that the angle between nonzero vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$ is

$$\theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right).$$

Thus, θ is acute ($0 < \theta < \pi/2$) if $\mathbf{u} \cdot \mathbf{v} > 0$, θ is obtuse ($\pi/2 < \theta < \pi$) if $\mathbf{u} \cdot \mathbf{v} < 0$ and $\theta = \pi/2$ if $\mathbf{u} \cdot \mathbf{v} = 0$.

Note that if $\mathbf{u} \cdot \mathbf{v} = 0$, the two vectors are perpendicular to each other. In general, this is a very special condition in linear algebra, so we give the following definition.

Definition 1.24. Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are called **orthogonal** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Remark. Notice that $\mathbf{u}, \mathbf{v}$ in the above definition does not necessarily have to be a nonzero vector. Hence, if either vector is a zero vector, then the two vectors are always orthogonal.

Example 1.25. Show that vectors $\mathbf{u} = (-1, 0, 2, 4)$ and $\mathbf{v} = (2, 1, 1, 0)$ are orthogonal.

Solution. Since

$$\mathbf{u} \cdot \mathbf{v} = (-1) \cdot 2 + 0 \cdot 1 + 2 \cdot 1 + 4 \cdot 0 = 0$$

we see that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

Theorem 1.26. Suppose $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ are standard unit vectors in $\mathbb{R}^n$. Then $\mathbf{e}_i$ and $\mathbf{e}_j$ are orthogonal for all i and j between 1 and n .

Proof. Use Theorem 1.16. ■

Orthogonality is special because of its unique properties, which are discussed in advanced topics of linear algebra. For now, we look at one of those properties: the orthogonal projection of vectors.

First, we need to discuss what projections are. Imagine a ball lying on the floor. If we install a huge light source on the top of the ball, a shadow will appear

on the floor, marking the regions where the light cannot reach. This shadow would be the projection of the ball due to the light.

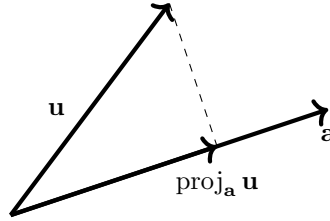


Figure 1.6: Orthogonal Projection of $\mathbf{u}$ on $\mathbf{a}$

Now let's go back to projections of vectors. Again, imagine a vector $\mathbf{u}$ lying on the coordinate plane. Suppose we were given another nonzero vector $\mathbf{a}$. Think that the line parallel to $\mathbf{a}$ is the surface, whereas $\mathbf{u}$ is the ball. Then, a huge light source installed above $\mathbf{u}$ would exert light moving in an orthogonal direction to $\mathbf{a}$, since $\mathbf{a}$ is the surface. And hence, it would create a shadow of $\mathbf{u}$, the projection of $\mathbf{u}$, on the surface, $\mathbf{a}$. We call it the **orthogonal projection** of $\mathbf{u}$ on $\mathbf{a}$, and denote it as $\text{proj}_{\mathbf{a}} \mathbf{u}$.

Notice that the orthogonal projection of $\mathbf{u}$ on $\mathbf{a}$ could change as the direction of $\mathbf{a}$ changes, since the light would come from a different angle. Also, the vector $\mathbf{u} - \text{proj}_{\mathbf{a}} \mathbf{u}$ seems to be orthogonal to $\mathbf{a}$, hence the name "orthogonal" projection.

So far, we've been discussing the intuitions behind orthogonal projections. Now we need to take a more rigorous stance on them. In the rigorous perspective, we deal with two problems:

- Does the orthogonal projection always exist?
- Is the orthogonal projection always unique?

which we refer to as the **existence problem** and the **uniqueness problem**, a common type of question that occurs in mathematics.

The following theorem gives answers to both problems.

Theorem 1.27. *If $\mathbf{u}$ and $\mathbf{a}$ are vectors in $\mathbb{R}^n$ where $\mathbf{a} \neq \mathbf{0}$, then there uniquely exists $\mathbf{u}_1$ and $\mathbf{u}_2$ in $\mathbb{R}^n$ such that*

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$$

where $\mathbf{u}_1$ is parallel to $\mathbf{a}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{a}$.

Proof. Suppose that

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 = \mathbf{v}_1 + \mathbf{v}_2$$

where $\mathbf{u}_1, \mathbf{v}_1$ are parallel to $\mathbf{a}$ and $\mathbf{u}_2, \mathbf{v}_2$ are orthogonal to $\mathbf{a}$. Then,

$$\mathbf{u}_1 - \mathbf{v}_1 = \mathbf{v}_2 - \mathbf{u}_2.$$

Also, let $\mathbf{u}_1 = k\mathbf{a}$ and $\mathbf{v}_1 = l\mathbf{a}$, since they are parallel to $\mathbf{a}$. Then, $(\mathbf{v}_2 - \mathbf{u}_2) \cdot \mathbf{a} = 0$ gives

$$\begin{aligned}(\mathbf{u}_1 - \mathbf{v}_1) \cdot \mathbf{a} &= \{(k - l)\mathbf{a}\} \cdot \mathbf{a} \\ &= (k - l) \|\mathbf{a}\|^2 = 0\end{aligned}$$

and $k - l = 0$, since $\mathbf{a}$ is a nonzero vector. Therefore, $\mathbf{u}_1 = \mathbf{v}_1$ and $\mathbf{u}_2 = \mathbf{v}_2$. ■

Hence, we now know that orthogonal projections always exist and are unique. In the following theorem, we find the formula for orthogonal projections when the vectors are given.

Theorem 1.28. *Let $\mathbf{u}$ and $\mathbf{a}$ be vectors in $\mathbb{R}^n$ where $\mathbf{a} \neq \mathbf{0}$. If $\mathbf{u}$ can be expressed as*

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 \tag{1.9}$$

where $\mathbf{u}_1$ is parallel to $\mathbf{a}$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{a}$, then

$$\mathbf{u}_1 = \text{proj}_{\mathbf{a}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \mathbf{a} \tag{1.10}$$

and

$$\mathbf{u}_2 = \mathbf{u} - \text{proj}_{\mathbf{a}} \mathbf{u} = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \mathbf{a}. \tag{1.11}$$

Proof. If θ is the angle between $\mathbf{u}$ and $\mathbf{a}$, then

$$\text{proj}_{\mathbf{a}} \mathbf{u} = (\|\mathbf{u}\| \cdot \cos \theta) \cdot \mathbf{e}_{\mathbf{a}} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|} \cdot \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \mathbf{a}.$$

(1.11) can be easily derived by substituting (1.10) into (1.9). ■

Turning a vector $\mathbf{u}$ into the sum of $\mathbf{u}_1$ and $\mathbf{u}_2$ that are orthogonal to each other is called **orthogonal decomposition**. Thus, (1.9) is an example of an orthogonal decomposition.

Consider (1.3) again, this time to notice that we decomposed the vector $\mathbf{u}$ into vectors $u_1\mathbf{e}_1, u_2\mathbf{e}_2, \dots, u_n\mathbf{e}_n$ which are all orthogonal to each other by Theorem 1.26. Hence, (1.3) is also an example of an orthogonal decomposition.

Example 1.29. *Given $\mathbf{u} = (1.5, 2)$ and $\mathbf{a} = (3, 1)$, express $\mathbf{u}$ as a sum of two vectors: one parallel to $\mathbf{a}$ and one orthogonal to $\mathbf{a}$.*

Solution. By (1.10),

$$\mathbf{u}_1 = \text{proj}_{\mathbf{a}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \mathbf{a} = \frac{1.5 \cdot 3 + 2 \cdot 1}{3^2 + 1^2} \mathbf{a} = 0.65\mathbf{a} = (1.95, 0.65)$$

and

$$\mathbf{u}_2 = \mathbf{u} - \text{proj}_{\mathbf{a}} \mathbf{u} = (1.5, 2) - (1.95, 0.65) = (-0.45, 1.35)$$

Hence, $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ is an orthogonal decomposition.

So far, we have seen how the inner product is used to find norms, distances, angles, and orthogonal projections of vectors. In the next section, we continue our discussion on geometric applications by looking at two geometric objects: lines and planes.

1.4 Lines and Planes in $\mathbb{R}^n$

We are already familiar with lines and planes in coordinate space, which can be expressed in linear equations. In this section, we want to examine a different way of expressing lines and planes, using vectors.

The usage of vectors allows us to analyze lines and planes in the coordinate space much more easily and with simplicity. It also helps us solve geometric problems with less difficulty.

Lines in $\mathbb{R}^2$

In $\mathbb{R}^2$, there are two ways to express a line with vectors.

Suppose there is a straight line l in $\mathbb{R}^2$. The **normal vector** of l is any vector $\mathbf{n}$ perpendicular to l . Following theorem shows how normal vectors are used to express a line in $\mathbb{R}^2$.

Theorem 1.30. *If l is a line in $\mathbb{R}^2$ that crosses $\mathbf{x}_0 = (x_0, y_0)$ and has a normal vector $\mathbf{n} = (a, b)$, then*

$$l = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0\}. \quad (1.12)$$

Proof. Notice that $\mathbf{x} - \mathbf{x}_0$ is parallel to l for all $\mathbf{x} \in l$, because $\mathbf{x}$ and $\mathbf{x}_0$ are both vectors in l . Since $\mathbf{n}$ is a normal vector, it must be orthogonal to all $\mathbf{x} - \mathbf{x}_0$. Hence, we have (1.12). ■

By (1.12), we also see that if $\mathbf{x} = (x, y) \in l$, then

$$a(x - x_0) + b(y - y_0) = 0.$$

This shows that the vector equation in (1.12) can be turned into a linear equation of x and y .

Another way of expressing a line is using parameters with vectors. The theorem below shows how it works.

Theorem 1.31. *If l is a line in $\mathbb{R}^2$ that crosses vector $\mathbf{x}_0 \in \mathbb{R}^2$ and is parallel to vector $\mathbf{v} \in \mathbb{R}^2$, then l can be expressed as*

$$l = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbf{x}_0 + t\mathbf{v}, t \in \mathbb{R}\}. \quad (1.13)$$

Proof. Let

$$l' = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = t\mathbf{v}, t \in \mathbb{R}\}.$$

Then l' is a line through the origin that is parallel to $\mathbf{v}$. Since, l is the result of translating l' by $\mathbf{x}_0$, we see that l is a line parallel to $\mathbf{v}$ and crossing $\mathbf{x}_0$. ■

This time, we used vector $\mathbf{v}$, which is parallel to the line. And we also used the parameter t to construct a straight line.

Notice that if $\mathbf{x}_0 = (x_0, y_0)$, $\mathbf{v} = (a, b)$, and $\mathbf{x} = (x, y)$ are in l , then (1.13) leads to

$$(x, y) = (x_0, y_0) + (a, b) \cdot t \quad \text{or} \quad \frac{x - x_0}{a} = \frac{y - y_0}{b} = t$$

for all $t \in \mathbb{R}$, which is again a linear equation of x and y .

Lines and Planes in $\mathbb{R}^3$

In $\mathbb{R}^3$, we deal with both lines and planes. We recall our analysis on lines in $\mathbb{R}^2$, and move on to the analysis of planes.

There were two ways to express a line in $\mathbb{R}^2$ using vectors. Recall Theorems 1.30 and 1.31. In $\mathbb{R}^3$, we can not use the method analogous to Theorem 1.30, since the direction of a normal vector for a line in $\mathbb{R}^3$ is not unique. However, we can still use the method in Theorem 1.31.

Theorem 1.32. *If l is a line in $\mathbb{R}^3$ that crosses $\mathbf{x}_0 = (x_0, y_0, z_0)$ and is parallel to $\mathbf{v} = (a, b, c)$, then l can be expressed as*

$$l = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \mathbf{x}_0 + t\mathbf{v}, t \in \mathbb{R}\}. \quad (1.14)$$

Proof. Similar to that of Theorem 1.31. ■

If $\mathbf{x} = (x, y, z) \in l$, then (1.14) gives

$$(x, y, z) = (x_0, y_0, z_0) + (a, b, c) \cdot t$$

or

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = t \quad (1.15)$$

for all $t \in \mathbb{R}$. Again, it is a linear equation of x, y, z .

Planes are not very much different from lines. Planes in $\mathbb{R}^3$ can be expressed in two ways, analogous to those used for lines in $\mathbb{R}^2$.

For a plane S in $\mathbb{R}^3$, the normal vector of S is a vector $\mathbf{n}$ that is perpendicular to the plane. The direction of $\mathbf{n}$ is uniquely determined, unlike lines in $\mathbb{R}^3$. Hence, normal vectors can be used again to express planes.

Theorem 1.33. *If S is a plane in $\mathbb{R}^3$ that crosses $\mathbf{x}_0 = (x_0, y_0, z_0)$ and has a normal vector $\mathbf{n} = (a, b, c)$, then*

$$S = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0\}. \quad (1.16)$$

Proof. If $\mathbf{x} \in S$, then the arrow of $\mathbf{x} - \mathbf{x}_0$ is on the plane. Since $\mathbf{n}$ is perpendicular to the plane, $\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$ for all $\mathbf{x} \in S$. ■

If $\mathbf{x} = (x, y, z) \in S$, then (1.16) leads to

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0, \quad (1.17)$$

a linear equation in a form distinct from (1.15). This is obvious, because a same equation can not express a line and a plane at the same time.

But why do equations (1.15) and (1.17) look fundamentally different? Consider the second method; using parameters to express a plane. Unlike lines, we need two parameters to do this. This is because a single parameter can only create a line, as we saw in Theorem 1.32. But a plane can not be uniquely determined by a single line. There needs to be at least two non-parallel lines to determine a single plane. Hence, we need two parameters and two non-collinear vectors determining the plane.

This fundamental difference between lines and planes leads to the difference in the form of the linear equations expressing them. Now refer to the following theorem to see how parameters are used to express planes.

Theorem 1.34. *If S is a plane in $\mathbb{R}^3$ that crosses vector $\mathbf{x}_0 \in \mathbb{R}^3$ and is parallel to non-collinear vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^3$, then S can be expressed by the equation*

$$S = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2, t_1, t_2 \in \mathbb{R}\}. \quad (1.18)$$

Proof. Let

$$S' = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2, t_1, t_2 \in \mathbb{R}\}.$$

Then, S' is a plane through the origin containing lines

$$l_1 = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = t_1\mathbf{v}_1, t_1 \in \mathbb{R}\} \quad \text{and} \quad l_2 = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = t_2\mathbf{v}_2, t_2 \in \mathbb{R}\}.$$

Since, S is the result of translating S' by the vector $\mathbf{x}_0$, then S becomes a plane crossing $\mathbf{x}_0$ and parallel to both $\mathbf{v}_1$ and $\mathbf{v}_2$. ■

If $\mathbf{x}_0 = (x_0, y_0, z_0)$, $\mathbf{v}_1 = (a_1, b_1, c_1)$, $\mathbf{v}_2 = (a_2, b_2, c_2)$ and $\mathbf{x} = (x, y, z) \in S$, then (1.18) gives

$$(x, y, z) = (x_0, y_0, z_0) + (a_1, b_1, c_1) \cdot t_1 + (a_2, b_2, c_2) \cdot t_2$$

and a system of equations

$$\begin{cases} a_1t_1 + a_2t_2 = x - x_0 \\ b_1t_1 + b_2t_2 = y - y_0 \\ c_1t_1 + c_2t_2 = z - z_0. \end{cases}$$

It is not easy to solve the above system since there are two parameters. But the system itself is already enough to fully express a plane.

In the next theorem, we see how vectors are used to find the distance between a point and a plane in $\mathbb{R}^3$.

Theorem 1.35. In $\mathbb{R}^3$, the distance between the point $P = (x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} \quad (1.19)$$

Proof. Let $Q = (x_1, y_1, z_1)$ be a point in the plane and $\mathbf{n} = (a, b, c)$. Then,

$$D = \left\| \text{proj}_{\mathbf{n}} \overrightarrow{QP} \right\| = \frac{|\mathbf{n} \cdot \overrightarrow{QP}|}{\|\mathbf{n}\|} = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}}$$

where $\overrightarrow{QP} = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$. Because Q is a point on the plane, we can use $ax_1 + by_1 + cz_1 + d = 0$ to get (1.19). ■

Lines and Planes in $\mathbb{R}^n$

We end this section with generalized definitions of lines and planes in $\mathbb{R}^n$.

Although we can not visualize $\mathbb{R}^n$ space where $n \geq 3$, we can still express lines and planes as a set of vectors. The following definitions of lines and planes in $\mathbb{R}^n$ are motivated from the cases of $\mathbb{R}^2$ and $\mathbb{R}^3$. Note that expressions using normal vectors is not appropriate for both lines and planes in $\mathbb{R}^n$ where $n \geq 3$. (Similar reason to why normal vectors can not be used for expressing lines in $\mathbb{R}^2$.)

Definition 1.36. If l is a line in $\mathbb{R}^n$ that crosses $\mathbf{x}_0 \in \mathbb{R}^n$ and is parallel to $\mathbf{v} \in \mathbb{R}^n$, then

$$l = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{x}_0 + t\mathbf{v}, t \in \mathbb{R}\}.$$

Definition 1.37. If S is a plane in $\mathbb{R}^n$ that crosses $\mathbf{x}_0 \in \mathbb{R}^n$ and is parallel to non-collinear vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^n$, then

$$S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2, t_1, t_2 \in \mathbb{R}\}.$$

1.5 Cross Product

In Section 1.3, we looked at the inner product, which was the product of two vectors that resulted in a scalar. Now we look at a special type of vector product that results in a vector.

Definition 1.38. If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in $\mathbb{R}^3$, then the **cross product** $\mathbf{u} \times \mathbf{v}$ is a vector in $\mathbb{R}^3$ defined as

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

Remark. The cross product is an operation defined only for vectors in $\mathbb{R}^3$.

Theorem 1.39. If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are vectors in $\mathbb{R}^3$, then:

(a) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}.$

$$(b) \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0.$$

$$(c) \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0.$$

$$(d) \|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta.$$

$$(e) \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$$

where θ is the angle between the vectors.

Proof. Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$.

(a) Then,

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) \\ &= -(v_2u_3 - v_3u_2, v_3u_1 - v_1u_3, v_1u_2 - v_2u_1) = -\mathbf{v} \times \mathbf{u}. \end{aligned}$$

(b) Using the distributive law of real numbers,

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) &= (u_1, u_2, u_3) \cdot (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) \\ &= u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) \\ &= v_1(u_2u_3 - u_3u_2) + v_2(-u_1u_3 + u_3u_1) + v_3(u_1u_2 - u_2u_1) = 0. \end{aligned}$$

(c) Similarly proven as (b).

(d) Rearranging the terms and using the properties of inner products, we have

$$\begin{aligned} \|\mathbf{u} \times \mathbf{v}\|^2 &= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\ &= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \\ &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 \\ &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 (1 - \cos^2 \theta) \\ &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \sin^2 \theta, \end{aligned}$$

which leads to $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$.

(e) Since $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, the vector $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ is on the plane formed by $\mathbf{v}$ and $\mathbf{w}$. Hence we can write

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = a\mathbf{v} + b\mathbf{w} \tag{1.20}$$

where a, b are constants. Then,

$$\mathbf{u} \cdot (\mathbf{u} \times (\mathbf{v} \times \mathbf{w})) = \mathbf{u} \cdot (a\mathbf{v} + b\mathbf{w}) \Rightarrow 0 = a(\mathbf{u} \cdot \mathbf{v}) + b(\mathbf{u} \cdot \mathbf{w}).$$

Now let

$$\frac{a}{\mathbf{u} \cdot \mathbf{w}} = -\frac{b}{\mathbf{u} \cdot \mathbf{v}} = \lambda$$

and substitute to (1.20) which leads to

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \lambda(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - \lambda(\mathbf{u} \cdot \mathbf{v})\mathbf{w}. \tag{1.21}$$

Finally, put $\mathbf{u} = (1, 0, 0)$, $\mathbf{v} = (0, 1, 0)$, $\mathbf{w} = (1, 0, 0)$ into (1.21) to get $\lambda = 1$. Therefore, (e) is proven. ■

Theorem 1.39 (a) indicates that the cross product is **anticommutative**. (b) and (c) imply that $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. Hence, if we are given two non-collinear vectors $\mathbf{u}, \mathbf{v}$, then a vector orthogonal to both vectors can be found by calculating the cross product $\mathbf{u} \times \mathbf{v}$. (d) implies that the size of the cross product $\mathbf{u} \times \mathbf{v}$ is equal to the area of a parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$. Also, it suggests an easy way to find the magnitude of $\mathbf{u} \times \mathbf{v}$. Finally, (e) is the formula for the **vector triple product**, which is often used in physics.

The below examples show how we can apply the properties of cross products to solve problems in geometry.

Example 1.40. Find the equation of a plane through the origin parallel to vectors $\mathbf{v}_1 = (1, 0, 2)$ and $\mathbf{v}_2 = (3, -1, 5)$.

Solution. The normal vector of the plane would be $\mathbf{v}_1 \times \mathbf{v}_2 = (2, 1, -1)$ since $\mathbf{v}_1 \times \mathbf{v}_2$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$. Hence, using the formula (1.17), we get $2x + y - z = 0$.

Example 1.41. Find the size of the angle between the planes $2x - 3y + z = 0$ and $4x + y - 2z = 0$.

Solution. The angle between the planes is equal to the angle between the normal vectors of the two planes. Since the normal vector of $2x - 3y + z = 0$ is $\mathbf{n}_1 = (2, -3, 1)$ and the normal vector of $4x + y - 2z = 0$ is $\mathbf{n}_2 = (4, 1, -2)$, the angle can be obtained by using (1.8). Hence,

$$\theta = \cos^{-1} \left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \right) = \cos^{-1} \left(\frac{3}{7\sqrt{6}} \right).$$

2 | Vector Spaces

In the previous chapter, we observed how mathematicians took off from the visual vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ and reached the vectors in $\mathbb{R}^n$. The key to this generalization was using coordinates. Considering vectors as purely algebraic objects allowed us to treat all ordered n -tuples as vectors.

Though we may suppose the $\mathbb{R}^n$ space to be a rather general set of vectors, mathematicians did not suffice with this generalization. They stated that as long as a certain object, whether it be a function or a sequence, satisfies certain algebraic properties, it can be treated as vectors in the most general sense. This was possible because algebraic objects are equivalent in identity as long as its algebraic properties are the same.

In this chapter, we examine this general notion of vectors. We will see what "vector spaces" means at the first half of the chapter, and see what "finite-dimensional" means at the second half of the chapter. By the end of this chapter, we will know what linear algebra mainly deals with at its core.

2.1 Vector Space Axioms

We begin our discussion with looking at what vectors are in the most general sense. In other words, we need to look at the basic algebraic properties, or axioms, that vectors must satisfy.

Definition 2.1. A *vector space*¹ is a triple $(V, +, \cdot)$, where V is a nonempty set of objects called **vectors**, $+: V \times V \rightarrow V$ is a binary operator called **addition**, and $\cdot: \mathbb{R} \times V \rightarrow V$ is a binary operator called **scalar multiplication**, that satisfies the following properties:

- (a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$.
- (b) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.
- (c) There exists $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$.

¹The vector space defined in Definition 2.1 is precisely called a **real vector space**, because the scalar used as the operand of scalar multiplication is restricted to real numbers. The scalar can actually be the element of any field, which includes real numbers and complex numbers too, but we do not consider this generality in this book.

(d) For all $\mathbf{u} \in V$ there exists $\mathbf{v} \in V$ such that $\mathbf{u} + \mathbf{v} = \mathbf{0}$.

(e) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ for all $\mathbf{u} \in V$ and $c, d \in \mathbb{R}$.

(f) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in V$ and $c \in \mathbb{R}$.

(g) $(cd)\mathbf{u} = c(d\mathbf{u})$ for all $\mathbf{u} \in V$ and $c, d \in \mathbb{R}$.

(h) $1\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$.

(a) and (b) represent the **commutative law** and the **associative law**, respectively. (c) implies that there exists an **additive identity** in V , and (d) implies that each vector has an **additive inverse**. (e) and (f) represent the **distributive law**, (g) represents the associative law between scalars and vectors. Finally, (h) implies that the scalar 1 is a **multiplicative identity**.

Since the ranges of $+$ and $\cdot$ are subsets of V , the result of both operations must always be in V . Such operations are called **closed** operations, because the operand and the result do not exceed the boundaries of the given set V .

Notice that the eight properties listed above are equivalent to the eight algebraic properties of vectors found in Theorem 1.7. We accept any algebraic object that satisfies these properties as vectors because they hold the same algebraic properties as vectors in $\mathbb{R}^n$ do. In the algebraic perspective, any operations made with general vectors are no different from operations made with vectors in $\mathbb{R}^n$. So although it may seem awkward to think that objects that aren't traditionally "vectors" can be treated as a vector, we have to admit that there isn't any practical problem with our definition.

Now we look at some examples of vector spaces that meet Definition 2.1.

Example 2.2. Consider the set $\{\mathbf{0}\}$, which consists only of the zero vector. It can be shown that if addition and scalar multiplication are defined on $\{\mathbf{0}\}$ as

$$\mathbf{0} + \mathbf{0} = \mathbf{0} \quad \text{and} \quad c\mathbf{0} = \mathbf{0}$$

for all $c \in \mathbb{R}$, then addition and scalar multiplication are closed, and properties in Definition 2.1 are all satisfied. Hence, $\{\mathbf{0}\}$ is a vector space, specially called the **zero vector space**.

Example 2.3. Consider the set $\mathbb{R}^n$, which was our main focus in Chapter 1. With addition and scalar multiplication defined as

$$(u_1, u_2, \dots, u_n) + (v_1, v_2, \dots, v_n) = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$$

$$c \cdot (u_1, u_2, \dots, u_n) = (cu_1, cu_2, \dots, cu_n),$$

we can check that $+$ and $\cdot$ are closed operations and $\mathbb{R}^n$ satisfies all properties in Definition 2.1. Hence, $\mathbb{R}^n$ is a vector space, and all of its elements are vectors. Specifically, we call $\mathbb{R}^n$ the n -dimensional **Euclidean vector space** or simply n -space.

Example 2.4. $\mathbb{P}_n$ denotes the set of all polynomials of order n or lower. By notation,

$$\mathbb{P}_n = \{c_0 + c_1t + \cdots + c_nt^n : c_0, c_1, \dots, c_n \in \mathbb{R}\}.$$

If addition and scalar multiplication are defined in the common sense, then we can check that $+$ and $\cdot$ are closed and $\mathbb{P}_n$ satisfies all properties in Definition 2.1. Hence, $\mathbb{P}_n$ is a vector space and all polynomials in $\mathbb{P}_n$ can be called vectors.

Example 2.5. $C(\mathbb{R})$ denotes the set of all real-valued functions continuous on $\mathbb{R}$. Given addition and scalar multiplication in its common sense, we can check that if $f, g \in C(\mathbb{R})$ and $c \in \mathbb{R}$, then $f + g \in C(\mathbb{R})$ and $cf \in C(\mathbb{R})$, which implies that $+$ and $\cdot$ are closed. Also, $C(\mathbb{R})$ meets all properties in Definition 2.1. Hence, $C(\mathbb{R})$ is a vector space and all continuous functions in $C(\mathbb{R})$ are vectors.

Example 2.6. Let S be the set of all sequences that map positive integers to some value in $\mathbb{R}$. If addition and scalar multiplication are defined as

$$\{a_n\} + \{b_n\} = \{a_n + b_n\} \quad \text{and} \quad c\{a_n\} = \{ca_n\},$$

then it can be shown that the operations are closed and satisfy all properties in Definition 2.1. Thus, S is also a vector space and all sequences in S are vectors.

Example 2.7. Consider the differential equation

$$\frac{d^2x}{dt^2} - \omega^2x = 0.$$

The solution set of this differential equation turns out to be

$$\{c_1e^{\omega x} + c_2e^{-\omega x} : c_1, c_2 \in \mathbb{R}\},$$

which is a vector space due to Definition 2.1.

As the above examples show, tuples, polynomials, functions, and sequences can also be vectors as long as the set satisfies all properties in Definition 2.1. From now on, whenever we use the term vector, we take these examples inclusively and understand that the term refers to a very general notion.

We now look at some common properties of vector spaces, which can be directly derived from the properties in Definition 2.1.

Theorem 2.8. The additive identity, or the zero vector $\mathbf{0}$, is unique for all vector space V .

Proof. Suppose $\mathbf{0}$ and $\mathbf{0}'$ are additive identity in vector space V . Hence,

$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u} \quad \text{and} \quad \mathbf{u} + \mathbf{0}' = \mathbf{0}' + \mathbf{u} = \mathbf{u}$$

for all $\mathbf{u} \in V$. Then,

$$\mathbf{0} = \mathbf{0} + \mathbf{0}' = \mathbf{0}',$$

which implies that the additive identity $\mathbf{0}$ is unique. ■

Theorem 2.9. *For all $\mathbf{u} \in V$, its additive inverse is unique.*

Proof. Let $\mathbf{u}$ be some vector in V , and suppose $\mathbf{v}$ and $\mathbf{v}'$ are additive inverses of $\mathbf{u}$ in vector space V . Hence,

$$\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{v}' = \mathbf{0}.$$

This leads to

$$\mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v} + (\mathbf{u} + \mathbf{v}') = (\mathbf{v} + \mathbf{u}) + \mathbf{v}' = \mathbf{0} + \mathbf{v}' = \mathbf{v}',$$

which implies that $\mathbf{v}$ is unique. ■

Since the additive inverse is unique for all vectors, we may denote the additive inverse of some vector $\mathbf{u} \in V$ as $-\mathbf{u}$.

Theorem 2.10. *If V is a vector space, then:*

- (a) $0\mathbf{u} = \mathbf{0}$ for all $\mathbf{u} \in V$.
- (b) $c\mathbf{0} = \mathbf{0}$ for all $c \in \mathbb{R}$.
- (c) $(-1)\mathbf{u} = -\mathbf{u}$ for all $\mathbf{u} \in V$ and $c \in \mathbb{R}$.

Proof. (a) We shall use the properties in Definition 2.1. First, we have

$$0\mathbf{u} + 0\mathbf{u} = (0 + 0)\mathbf{u} = 0\mathbf{u},$$

which leads to

$$[0\mathbf{u} + 0\mathbf{u}] + (-0\mathbf{u}) = 0\mathbf{u} + (-0\mathbf{u})$$

where $-0\mathbf{u}$ is the negative of $0\mathbf{u}$, and $0\mathbf{u} = \mathbf{0}$.

(b), (c) **Exercise.** ■

2.2 Subspaces

In this section, we discuss vector spaces that are subsets of a larger vector space, which we shall call subspaces. The following gives a precise definition of subspaces.

Definition 2.11. *Let V be a vector space. If W is a subset of V and is a vector space under the addition and scalar multiplication defined on V , then we call W a **subspace** of V , and denote $W \leq V$.*

Example 2.12. *If V is a vector space, then the zero vector space $\{\mathbf{0}\}$ is a subspace of V since $\mathbf{0} \in V$ and $\{\mathbf{0}\}$ is a vector space. Since the zero vector space can be a subspace for all vector spaces, it is also called a **trivial subspace**.*

By now you may have noticed that it is a troublesome task to show that a given set is a vector space, for we have to check all criteria in Definition 2.1. Fortunately,

in the discussion of subspaces, there are far less criteria that have to be checked to show that a certain set is a subspace of another set.

Suppose V is a vector space where its addition and scalar multiplication are defined. Since a subspace of V inherits the addition and scalar multiplication defined on V , we do not have to show properties like Definition 2.1 (a), (b), (e), (f), (g), and (h). Meanwhile, we still have to check that addition and scalar multiplication are closed, and that (c), (d) is satisfied, because we can not ensure that these properties remain satisfied when the set V is restricted to a smaller set W .

In the below theorem, we shall see that showing three of the properties above suffices to claim that W is a subspace of V . It turns out that Definition 2.1 does not have to be separately shown.

Theorem 2.13 (Subspace Test). *Let V be a vector space and W be a subset of V under the addition and scalar multiplication defined on V . Then W is a subspace of V if and only if W satisfies the following three properties:*

- (a) $\mathbf{u} + \mathbf{v} \in W$ for all $\mathbf{u}, \mathbf{v} \in W$.
- (b) $c\mathbf{u} \in W$ for all $\mathbf{u} \in W$ and $c \in \mathbb{R}$.
- (c) There exists $\mathbf{0} \in W$ such that $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{0}$ for all $\mathbf{u} \in W$.

Proof. ($\Rightarrow$) If W is a subspace of V , then W is a vector space that satisfies Definition 2.1 Hence, properties (a), (b), (c) all hold for W .

($\Leftarrow$) Suppose (a), (b), (c) all hold for W . If $\mathbf{u}$ is a vector in W , then by (b), $(-1)\mathbf{u}$ is also in W . Since scalar multiplication is defined over vector space V , we can use Theorem 2.10 (c), which implies that $(-1)\mathbf{u} = -\mathbf{u}$ is in W . Hence, all additive inverses exist within W , allowing W to meet all conditions laid in Definition 2.1. Therefore, W is a vector space. ■

Now we look at some examples of subspaces.

Example 2.14. Recall that $\mathbb{R}^n$ is a vector space. Consider

$$l = \{(u_1, u_2) : u_1 = 0, u_2 \in \mathbb{R}\},$$

a subset of $\mathbb{R}^2$. Using Theorem 2.13, it can be shown that l is a vector space and thus, l is a subspace of $\mathbb{R}^2$.

Notice that geometrically, l is a line passing through the origin above $\mathbb{R}^2$ plane. In fact, all lines passing through the origin are a subspace of $\mathbb{R}^2$, as well as the zero vector space $\{\mathbf{0}\}$ and $\mathbb{R}^2$ itself.

For $\mathbb{R}^3$, any line or plane that passes the origin can be a subspace of $\mathbb{R}^3$, as well as $\{\mathbf{0}\}$ and $\mathbb{R}^3$. All linear structures (lines, planes, hyperplanes, ...) in $\mathbb{R}^n$ that include the zero vector (pass through the origin) turn out to be a subspace of $\mathbb{R}^n$.

Example 2.15. Recall $\mathbb{P}_n$, the set of all polynomials of order n or below, a vector space. Consider S , a subset of $\mathbb{P}_n$, defined as

$$S = \{c_0 + c_1t + \cdots + c_nt^n : c_0 + c_1 + \cdots + c_n = 0\}.$$

Let $u(t) = u_0 + u_1t + \cdots + u_nt^n, v(t) = v_0 + v_1t + \cdots + v_nt^n$ be polynomials in S . Then, the polynomial

$$(u + v)(t) = u(t) + v(t) = (u_0 + v_0) + (u_1 + v_1)t + \cdots + (u_n + v_n)t^n$$

is a polynomial in S since $(u_0 + u_1 + \cdots + u_n) + (v_0 + v_1 + \cdots + v_n) = 0$. Also, the polynomial

$$(cu)(t) = cu(t) = cu_0 + cu_1t + \cdots + cu_nt^n$$

is a polynomial in S since $cu_0 + cu_1 + \cdots + cu_n = c(u_0 + u_1 + \cdots + u_n) = 0$. Therefore, S is closed under addition and scalar multiplication and has the additive inverse $\mathbf{0}$. Hence, by Theorem 2.13, S is a subspace of $\mathbb{P}_n$.

The following example is the case where the subset of a vector space doesn't turn out to be a vector space.

Example 2.16. Consider

$$l' = \{(u_1, u_2) : u_1 + u_2 = 1, u_1, u_2 \in \mathbb{R}\},$$

a subset of vector space $\mathbb{R}^2$. We can check that l' is not a vector space since l' does not include the additive identity $\mathbf{0} = (0, 0)$.

Geometrically, l' is a line on $\mathbb{R}^2$ that does not pass through the origin. Hence, we can visually see that l' is not a vector space too.

l' can also be expressed as $\{(0, 1) + (1, -1)u_1 : u_1 \in \mathbb{R}\}$, which implies that l' is a line parallel to the vector $(1, -1)$ translated from the origin by $(0, 1)$. Vector spaces that are translated by a vector, like l' , are called **Affine spaces**.

Intersections and Sums

We now discuss some operations between the subspaces that result in another subspace. First is an operation we are already familiar with - the intersection of sets.

Theorem 2.17. If $W_1, \dots, W_n$ are subspaces of a vector space V , then their intersection

$$W = \bigcap_{i=1}^n W_i$$

is also a subspace of V .

Proof. First, W is a subset of V , since $W_1, \dots, W_n$ are subspaces of V . Claim that the three properties in Theorem 2.13 hold for W . First, if $\mathbf{u}, \mathbf{v}$ are in W , then $\mathbf{u}, \mathbf{v}$ are in $W_1, \dots, W_n$. Since $W_1, \dots, W_n$ are vector spaces, $\mathbf{u} + \mathbf{v}$ is also in all

of $W_1, \dots, W_n$. Hence, $\mathbf{u} + \mathbf{v}$ is in W . Similarly, if $\mathbf{u}$ is in W , then for some scalar $c \in \mathbb{R}$, $c\mathbf{u}$ is in W too. Finally, since the zero vector $\mathbf{0}$ exists in all of $W_1, \dots, W_n$, it also exists in W . Therefore, W is a subspace of V . ■

Now we define a new operation - the sum between sets. We also examine the theorems that result from this new operation.

Definition 2.18. Suppose $W_1, \dots, W_n$ are subsets of V . The **sum** of $W_1, \dots, W_n$, denoted as $W_1 + \dots + W_n$, is a set defined as

$$W_1 + \dots + W_n = \{\mathbf{w}_1 + \dots + \mathbf{w}_n : \mathbf{w}_1 \in W_1, \dots, \mathbf{w}_n \in W_n\}.$$

Theorem 2.19. If $W_1, \dots, W_n$ are subspaces of vector space V , then the sum $W_1 + \dots + W_n$ is the smallest subspace of V containing $W_1, \dots, W_n$.

Proof. Let $W = W_1 + \dots + W_n$. First, claim that W is a subspace of V . Due to Theorem 2.13, it suffices to show that (i) W is closed under addition, (ii) closed under scalar multiplication, and (iii) has the additive identity $\mathbf{0}$.

If $\mathbf{u}, \mathbf{v} \in W$, then there exists $u_1, v_1 \in W_1, \dots, u_n, v_n \in W_n$ such that

$$\mathbf{u} = u_1 + \dots + u_n \quad \text{and} \quad \mathbf{v} = v_1 + \dots + v_n.$$

Hence,

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (u_1 + \dots + u_n) + (v_1 + \dots + v_n) \\ &= (u_1 + v_1) + \dots + (u_n + v_n), \end{aligned}$$

where $u_1 + v_1 \in W_1, \dots, u_n + v_n \in W_n$, which implies that $\mathbf{u} + \mathbf{v} \in W$. Similarly, $c\mathbf{u} \in W$ for all scalar c , and finally, $\mathbf{0} \in W$ since $W_1, \dots, W_n$ are all vector spaces.

Next, show that W is the smallest subspace of V containing $W_1, \dots, W_n$. We can check that $W_1 \subset W, \dots, W_n \subset W$, which implies that W contains $W_1, \dots, W_n$. Also check that all subspaces that contain $W_1, \dots, W_n$ must at least include all elements of W , since the subspace must be closed under addition and scalar multiplication. Hence, W is the smallest subspace of V containing $W_1, \dots, W_n$. ■

From Definition 2.18, we can claim that if $W_1, \dots, W_n$ are subspaces of V , then all elements in the sum $W_1 + \dots + W_n$ can be expressed as the sum of elements each from $W_1, \dots, W_n$. In other words, for all vector $\mathbf{w} \in W$, there exists $\mathbf{w}_1 \in W_1, \dots, \mathbf{w}_n \in W_n$ such that $\mathbf{w} = \mathbf{w}_1 + \dots + \mathbf{w}_n$.

We may also consider a special case of the sum, where we can guarantee that for all vectors $\mathbf{w} \in W$, there uniquely exists $\mathbf{w}_1 \in W_1, \dots, \mathbf{w}_n \in W_n$ such that $\mathbf{w}$ is the sum of $\mathbf{w}_1, \dots, \mathbf{w}_n$, which implies that there is a bijection between $\mathbf{w}$ and $(\mathbf{w}_1, \dots, \mathbf{w}_n)$.

Definition 2.20. Suppose $W_1, \dots, W_n$ are subspaces of vector space V . If all elements of the sum $W_1 + \dots + W_n$ can be uniquely expressed as the sum of the elements of $W_1, \dots, W_n$, then we specially call the sum a **direct sum**, and denote it as $W_1 \oplus \dots \oplus W_n$.

It is difficult to check that a given sum is a direct sum using the above definition. Hence, in the next theorem, we give an easier criteria used to show that a sum is a direct sum.

Theorem 2.21. *Suppose W_1, W_2 are subspaces of vector space V . Then, $W_1 + W_2$ is a direct sum if and only if $W_1 \cap W_2 = \{\mathbf{0}\}$.*

Proof. ($\Rightarrow$) Suppose $W_1 + W_2$ is a direct sum, which implies that for all $\mathbf{u}$ in $W_1 + W_2$, there uniquely exists $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W_2$ such that $\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{u}$.

Suppose $W_1 \cap W_2 \neq \{\mathbf{0}\}$. Hence there exists nonzero vector $\mathbf{v}$ such that $\mathbf{v} \in W_1 \cap W_2$. Therefore, $\mathbf{v} \in W_1$ and $\mathbf{v} \in W_2$, which implies that $\mathbf{v} \in W_1 + W_2$ does not have a unique representation, a contradiction.

($\Leftarrow$) Suppose $\mathbf{v}$ is a vector in $W_1 + W_2$ and

$$\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}'_1 + \mathbf{w}'_2,$$

where $\mathbf{w}_1, \mathbf{w}'_1$ are vectors in W_1 and $\mathbf{w}_2, \mathbf{w}'_2$ are vectors in W_2 . Since

$$\mathbf{w}_1 - \mathbf{w}'_1 = \mathbf{w}'_2 - \mathbf{w}_2,$$

where $\mathbf{w}_1 - \mathbf{w}'_1 \in W_1$ and $\mathbf{w}'_2 - \mathbf{w}_2 \in W_2$, both $\mathbf{w}_1 - \mathbf{w}'_1$ and $\mathbf{w}'_2 - \mathbf{w}_2$ are vectors in $W_1 \cap W_2 = \{\mathbf{0}\}$. Therefore, $\mathbf{w}_1 = \mathbf{w}'_1$ and $\mathbf{w}_2 = \mathbf{w}'_2$, which gives that $W_1 + W_2$ is a direct sum. ■

Theorem 2.22. *Suppose $W_1, \dots, W_n$ are subspaces of vector space V . Then, $W_1 + \dots + W_n$ is a direct sum if and only if $W_i \cap W_j = \{\mathbf{0}\}$ for all integers $i \neq j$ between 1 and n .*

Proof. Use mathematical induction on n . If $n = 2$, then $W_1 + W_2$ is a direct sum if and only if $W_1 \cap W_2 = \{\mathbf{0}\}$ by Theorem 2.21.

Now suppose that $W_1 + \dots + W_k$ is a direct sum if and only if $W_i \cap W_j = \{\mathbf{0}\}$ for all integers $i \neq j$ between 1 and $n - 1$. Then, $(W_1 + \dots + W_k) + W_{k+1}$ is a direct sum if and only if

$$(W_1 + \dots + W_k) \cap W_{k+1} = \{\mathbf{0}\}.$$

This implies that $W_i \cap W_{k+1} = \{\mathbf{0}\}$ for all integers i between 1 and k . Therefore, $W_i \cap W_j = \{\mathbf{0}\}$ for all integers $i \neq j$ between 1 and $k + 1$, proving the theorem. ■

Hence, vectors in $W_1 + \dots + W_n$ can be uniquely decomposed into vectors in each set $W_1, \dots, W_n$ when all distinct pairs of $W_1, \dots, W_n$ have intersection $\{\mathbf{0}\}$.

In most contexts of linear algebra where vector spaces are involved, we often refer to subspaces of vector spaces rather than the "parent" vector space, because subspaces are more special and unique in character. Hence, the discussions made in this section is expected to appear repeatedly throughout the book.

2.3 Span and Linear Independence

Starting from this section, we will be building on two fundamental theorems of vector spaces. First is the linkage between vector spaces and the $\mathbb{R}^n$ space. Second is the justification to defining the "dimension" of vector spaces.

But prior to examining these two theorems, we observe two important conditions that a set of vectors can satisfy: span and linear independence. Span is the question of whether we have enough vectors to represent all vectors in a vector space; it guarantees the existence of representation. Linear independence is the question of whether the vectors in the set can not represent each other; it guarantees the uniqueness of representation.

A firm understanding of these two conditions will serve to better our understanding on the structure of vector spaces. This will be our focus for the latter part of this chapter.

Span

We begin with defining the linear combination of vectors, which are vectors that are formed by applying scalar multiplication and addition to the given vectors.

Definition 2.23. For vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ in vector space V , its **linear combination** is a vector in V of the form $c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$ where $c_1, \dots, c_n$ are scalars in $\mathbb{R}$.

Notice that all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_n$ in V are also vectors in V , because addition and scalar multiplication are closed under V .

Now we define the span of a set of vectors, which is the set of all linear combinations of the given vectors.

Definition 2.24. If $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a set of vectors in vector space V , then the **span** of S is denoted as $\text{span}(S)$ and defined as

$$\text{span}(S) = \{c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n : c_1, \dots, c_n \in \mathbb{R}\}.$$

If $W = \text{span}(S)$, then we say that S spans W , and call S the spanning set of W .

Example 2.25. Consider the standard unit vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ in $\mathbb{R}^n$. Since

$$\mathbf{e}_1 = (1, 0, \dots, 0), \mathbf{e}_2 = (0, 1, \dots, 0), \dots, \mathbf{e}_n = (0, 0, \dots, 1),$$

the span of $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ will be

$$\begin{aligned} \text{span}\{\mathbf{e}_1, \dots, \mathbf{e}_n\} &= \{c_1\mathbf{e}_1 + \dots + c_n\mathbf{e}_n : c_1, \dots, c_n \in \mathbb{R}\} \\ &= \{(c_1, \dots, c_n) : c_1, \dots, c_n \in \mathbb{R}\} = \mathbb{R}^n. \end{aligned}$$

Example 2.26. Recall from Chapter 1 that line l in $\mathbb{R}^n$ crossing the origin and parallel to $\mathbf{v} \in \mathbb{R}^n$ is defined as

$$l = \{\mathbf{x} : \mathbf{x} = t\mathbf{v}, t \in \mathbb{R}\}.$$

Notice that $l = \text{span}\{\mathbf{v}\}$, since l is the set of all linear combinations of $\mathbf{v}$.

Again, recall that plane S in $\mathbb{R}^n$ crossing the origin and parallel to non-collinear vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^n$ is defined as

$$S = \{\mathbf{x} : \mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2, t_1, t_2 \in \mathbb{R}\}.$$

This time, $S = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, since S is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2$.

The span of vectors in $\mathbb{R}^n$ are represented as linear structures like lines and planes that include the vectors in the given set. This is because the span is always a subspace, and the subspace of $\mathbb{R}^n$ is always a linear structure passing the origin.

The span turns out to hold special properties that lead us to important contexts. The following theorems present these properties.

Theorem 2.27. *If $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a set of vectors in vector space V , then $\text{span}(S)$ is a subspace of V .*

Proof. Let $W = \text{span}(S)$. If $\mathbf{u} = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$ and $\mathbf{v} = d_1\mathbf{v}_1 + \dots + d_n\mathbf{v}_n$, then

$$\mathbf{u} + \mathbf{v} = (c_1 + d_1)\mathbf{v}_1 + \dots + (c_n + d_n)\mathbf{v}_n$$

is a vector in W , hence W is closed under addition. Also, for some scalar $k \in \mathbb{R}$,

$$k\mathbf{u} = kc_1\mathbf{v}_1 + \dots + kc_n\mathbf{v}_n$$

is again a vector in W , hence W is closed under scalar multiplication too. Finally, $\mathbf{0}$ is in W since $\mathbf{0} = 0\mathbf{v}_1 + \dots + 0\mathbf{v}_n$. Therefore, by Theorem 2.13, W is a subspace of V . ■

Theorem 2.28. *If $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a set of vectors in vector space V , then $\text{span}(S)$ is the smallest subspace of V containing S . In other words, if $\{W_\alpha\}_{\alpha \in A}$ is an indexed collection of all subspaces of V containing S , then*

$$\text{span}(S) = \bigcap_{\alpha \in A} W_\alpha.$$

Proof. Let W be some subspace of V containing S . Since W is a vector space, it must contain $\text{span}(S)$ for W to be closed under addition and scalar multiplication. Therefore, the intersection of all subspaces of V containing S is $\text{span}(S)$. ■

Theorems 2.27 and 2.28 together imply that $\text{span}(S)$ is the "least requirement" for a subspace of V including S . This is the meaning of span in the perspective of subspaces.

Heading back to the definition of the span of S , we obtain a fundamental result, given in the below theorem.

Theorem 2.29. *If $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a set in vector space V that S spans W , a subspace of V , then there exists $c_1, \dots, c_n \in \mathbb{R}$ such that $\mathbf{w} = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$ for all $\mathbf{w} \in W$.*

If S spans W , we are guaranteed that all vectors in W can be expressed as the linear combination of S . In other words, we are guaranteed the existence of representation using S for all vectors in W . This is the meaning of span in the perspective of representation.

Linear Independence

Suppose $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a set of vectors in vector space V , and $\text{span}(S) = W$. If $\mathbf{v}_{n+1}$ is a vector that can be expressed as a linear combination of S , then the span of $S \cup \{\mathbf{v}_{n+1}\}$ will be equivalent to W , since the linear combination of $S \cup \{\mathbf{v}_{n+1}\}$ can be expressed as a linear combination of S if we substitute the terms of $\mathbf{v}_1, \dots, \mathbf{v}_n$ into $\mathbf{v}_{n+1}$.

If we were told to select some vectors in W to construct a set that spans W , it would be unnecessary to select and add $\mathbf{v}_{n+1}$ to S , since the resulting span does not change. $\mathbf{v}_{n+1}$ is an unnecessary vector, and $S \cup \{\mathbf{v}_{n+1}\}$ includes unnecessary vectors.

Linear independence determines whether the given set consists of such unnecessary vectors. Because we want to minimize given information as much as possible, we want the set to be linearly independent. The following definition provides the criterion that a set of vectors has to meet in order to be linearly independent.

Definition 2.30. A set of vectors $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ in vector space V is called **linearly dependent** if there exists scalars $c_1, \dots, c_n \in \mathbb{R}$ that are not all zero such that

$$c_1 \mathbf{v}_1 + \dots + c_n \mathbf{v}_n = \mathbf{0}. \quad (2.1)$$

Otherwise, S is called **linearly independent**. So if S is linearly independent, then the only solution of (2.1) is $(c_1, \dots, c_n) = (0, \dots, 0)$.

In the next theorem, we check that the criterion above is an equivalent statement to that vectors in S can be represented as a linear combination of other vectors in S .

Theorem 2.31. Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a set of vectors in vector space V . Then the following are equivalent:

- (a) S is linearly dependent.
- (b) There exists $i \in \{1, \dots, n\}$ such that $\mathbf{v}_i \in \text{span}(S \setminus \{\mathbf{v}_i\})$.
- (c) There exists $i \in \{1, \dots, n\}$ such that $\text{span}(S) = \text{span}(S \setminus \{\mathbf{v}_i\})$.

Proof. (a) $\Rightarrow$ (b) Since S is linearly dependent, there exists $c_1, \dots, c_n \in \mathbb{R}$ not all zero such that $c_1 \mathbf{v}_1 + \dots + c_n \mathbf{v}_n = \mathbf{0}$. Without loss of generality, suppose $c_i \neq 0$. Then,

$$\mathbf{v}_i = -\frac{c_1}{c_i} \mathbf{v}_1 - \dots - \frac{c_{i-1}}{c_i} \mathbf{v}_{i-1} - \frac{c_{i+1}}{c_i} \mathbf{v}_{i+1} - \dots - \frac{c_n}{c_i} \mathbf{v}_n,$$

which implies that $\mathbf{v}_i$ can be expressed as a linear combination of $S \setminus \{\mathbf{v}_i\}$. In other words, $\mathbf{v}_i \in \text{span}(S \setminus \{\mathbf{v}_i\})$.

(b) $\Rightarrow$ (c) Let $W_1 = \text{span}(S)$ and $W_2 = \text{span}(S \setminus \{\mathbf{v}_i\})$. Suppose i is an integer in $\{1, \dots, n\}$ and $\mathbf{v}_i$ is a vector in $\text{span}(S \setminus \{\mathbf{v}_i\})$. Then, there exists scalars $c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n \in \mathbb{R}$ such that

$$\mathbf{v}_i = c_1 \mathbf{v}_1 + \dots + c_{i-1} \mathbf{v}_{i-1} + c_{i+1} \mathbf{v}_{i+1} + \dots + c_n \mathbf{v}_n. \quad (2.2)$$

It is obvious that W_2 is a subset of W_1 . If $\mathbf{v} \in W_1$, then there exists $d_1, \dots, d_n$ such that

$$\mathbf{v} = d_1 \mathbf{v}_1 + \dots + d_n \mathbf{v}_n. \quad (2.3)$$

Substituting (2.2) into (2.3) gives that $\mathbf{v} \in W_2$. Hence, W_1 is a subset of W_2 and finally, $W_1 = W_2$.

(c) $\Rightarrow$ (a) Suppose $\text{span}(S) = \text{span}(S \setminus \{\mathbf{v}_i\})$. Then, there exists $\mathbf{v} \in \text{span}(S)$ such that

$$\begin{aligned} \mathbf{v} &= c_1 \mathbf{v}_1 + \dots + c_n \mathbf{v}_n \\ &= d_1 \mathbf{v}_1 + \dots + d_{i-1} \mathbf{v}_{i-1} + d_{i+1} \mathbf{v}_{i+1} + \dots + d_n \mathbf{v}_n \end{aligned}$$

where $c_1, \dots, c_n, d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_n \in \mathbb{R}$ and $c_i \neq 0$. Hence,

$$\mathbf{0} = (c_1 - d_1) \mathbf{v}_1 + \dots + c_i \mathbf{v}_i + \dots + (c_n - d_n) \mathbf{v}_n,$$

where $c_1 - d_1, \dots, c_i, \dots, c_n - d_n$ are not all zero. Therefore, S is linearly dependent. ■

The equivalence of (a) and (b) in the above statement suggests that the criterion of linear independence given in Definition 2.30 allows linear independence to be the criterion that determines whether a set involves of any unnecessary vectors.

Also, the equivalence of (b) and (c) suggests that the vector $\mathbf{v}_i$, which can be expressed as the linear combination of S , is an unnecessary vector since adding it to or deleting it from S does not affect $\text{span}(S)$. The following example is an illustration of this theorem on $\mathbb{R}^3$ space.

Example 2.32. Suppose $\mathbf{v}_1 = (1, 0, 2)$, $\mathbf{v}_2 = (0, -3, 1)$, $\mathbf{v}_3 = (2, 9, 1)$. Since $\mathbf{v}_3 = 2\mathbf{v}_1 + (-3)\mathbf{v}_2$, we can check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. Hence, $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Again, we can check that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linear independent. Hence, removing any vector from $\{\mathbf{v}_1, \mathbf{v}_2\}$ would lead to a different result for its span.

Finally, the span of $\{\mathbf{v}_1, \mathbf{v}_2\}$ would be a plane in $\mathbb{R}^3$ determined by the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, passing the origin. Notice that the plane would also include $\mathbf{v}_3$, which is why adding $\mathbf{v}_3$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$ does not affect the span of the set.

If S is a linearly independent set of vectors in vector space V , then all vectors in $\text{span}(S)$ can be expressed uniquely as the linear combination of S . Whereas span guaranteed the existence of representation, linear independence guarantees the uniqueness of representation.

This is because if S was linearly dependent, we could express some vector $\mathbf{v}_i \in S$ as a linear combination of other vectors in S and substitute that equation to obtain a different representation of some vector in $\text{span}(S)$.

But note that linear independence itself can not guarantee existence. If $V \neq \text{span}(S)$, then there exists $\mathbf{v} \in V$ such that $\mathbf{v} \notin \text{span}(S)$ (Why?), which means that $\mathbf{v}$ can not be expressed as the linear combination of S . Hence, linear independence alone would mean that some vector in V either has no representation or a unique representation.

2.4 Basis and Dimension

Now that we have all the cards laid out, it is time to examine the two fundamental theorems of vector spaces. We begin with looking at the relationship between vector spaces and the $\mathbb{R}^n$ space.

Basis and Coordinate Vectors

Recall that span guarantees existence and linear independence guarantees uniqueness. Two conditions combined, we have both existence and uniqueness guaranteed. Hence we present the following definition and theorems.

Definition 2.33. A set of vectors $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ in vector space V is called a **basis** for V if $\mathcal{B}$ is linearly independent and spans V .

Example 2.34. The standard unit vectors of $\mathbb{R}^n$, $\mathbf{e}_1, \dots, \mathbf{e}_n$, span $\mathbb{R}^n$ and are linearly independent. Hence, $\mathcal{E} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$. More specifically, we call $\mathcal{E}$ the **standard basis** for $\mathbb{R}^n$.

The basis of a vector space is not necessarily unique. In $\mathbb{R}^2$, the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{v}_1 = (2, -1)$ and $\mathbf{v}_2 = (0, 3)$, is also a basis for $\mathbb{R}^2$.

Example 2.35. The standard basis for $\mathbb{P}_n$ is $\mathcal{E} = \{1, t, \dots, t^n\}$. Check that $\mathcal{E}$ is linearly independent and spans $\mathbb{P}_n$.

Example 2.36. Recall that the solution set of the differential equation

$$\frac{d^2x}{dt^2} - \omega^2x = 0$$

is

$$\{c_1 e^{\omega x} + c_2 e^{-\omega x} : c_1, c_2 \in \mathbb{R}\}.$$

A basis for the above vector space can be $\mathcal{B} = \{e^{\omega x}, e^{-\omega x}\}$, since $\mathcal{B}$ spans the solution set and is linearly independent.

Remark. We have not yet discussed when a set of functions in a vector space can be linearly independent. Although it follows the original definition of linear independence, we use a different tool called the **Wronskian** to determine whether the set is linearly independent or not. Specifics are not dealt in this book, as it is to be dealt in a course on differential equations.

Theorem 2.37. *If $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for vector space V , then for all vector $\mathbf{v} \in V$, there uniquely exists scalars $c_1, \dots, c_n \in \mathbb{R}$ such that*

$$\mathbf{v} = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n.$$

Proof. Since $\text{span}(\mathcal{B}) = V$, we have guarantee that all vectors $\mathbf{v} \in V$ can be represented as the linear combination of vectors in $\mathcal{B}$. Hence, suppose that there exists $c_1, \dots, c_n \in \mathbb{R}$ and $d_1, \dots, d_n \in \mathbb{R}$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n = d_1\mathbf{v}_1 + \dots + d_n\mathbf{v}_n.$$

This gives us

$$\mathbf{0} = (c_1 - d_1)\mathbf{v}_1 + \dots + (c_n - d_n)\mathbf{v}_n,$$

and since $\mathcal{B}$ is linearly independent, $c_1 - d_1, \dots, c_n - d_n$ must all be equal to 0. Therefore, $c_1 = d_1, \dots, c_n = d_n$. ■

Notice that in the above proof, we used the condition of span to assume the existence of representation and used the condition of linear independence to show that such representation is unique. Hence, the two conditions combined together, the basis representation for all vectors in V is unique. This gives justification to the following definition.

Definition 2.38. *Let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for vector space V . If $\mathbf{v}$ is a vector in V that can be expressed as $\mathbf{v} = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$, where $c_1, \dots, c_n \in \mathbb{R}$, then the vector $[\mathbf{v}]_{\mathcal{B}} = (c_1, \dots, c_n)$ in $\mathbb{R}^n$ is called the **coordinate vector** of $\mathbf{v}$ with respect to basis $\mathcal{B}$.*

By Theorem 2.37, a mapping between V and $\mathbb{R}^n$ that maps $\mathbf{v}$ into $[\mathbf{v}]_{\mathcal{B}}$ is bijective. Hence, we can use vectors in $\mathbb{R}^n$ to express any abstract vectors in V . This is the first fundamental theorem of vector spaces. It implies that all vector spaces with a basis of n vectors have the same structure as $\mathbb{R}^n$.

Example 2.39. *Given vectors $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (-1, 0, 3)$, $\mathbf{v}_3 = (3, -1, 1)$, we can check that $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$. Hence, all vectors $\mathbf{v} = (x, y, z)$ in $\mathbb{R}^3$ can be uniquely represented as*

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3.$$

To find the coordinate vector $[\mathbf{v}]_{\mathcal{B}} = (c_1, c_2, c_3)$, we have to solve the system of equations

$$\begin{cases} c_1 - c_2 + 3c_3 = x \\ 2c_1 - c_3 = y \\ c_1 + 3c_2 + c_3 = z. \end{cases}$$

where x, y, z are given constants. We will deal with solving the above system of equations in Chapter 3, so we skip this process for now.

Dimension

Next, we look into the second fundamental theorem of vector spaces: the justification to defining the "dimension". Before we dive deeper, we first divide vector spaces into the following two categories:

Definition 2.40. *A vector space is called **finite-dimensional** if it can be spanned by a finite set of vectors. Otherwise, it is called **infinite-dimensional**.*

Remark. *From now on, we shall only discuss finite-dimensional vector spaces, since understanding infinite-dimensional vector spaces requires extensive knowledge on analysis. Infinite-dimensional vector spaces are dealt in a separate branch of mathematics called **functional analysis**.*

Recall that the basis of a vector space is not unique. In fact, as we have examined, $\mathbb{R}^3$ may hold infinitely many bases. But these infinitely many bases share a common characteristic - their cardinality. This is our second fundamental theorem. Now we focus on developing a formal proof for this fact.

We begin with the following lemma that provides a useful idea for proving that all bases have the same cardinality.

Lemma 2.41. *Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a set of vectors in vector space V . If $\mathbf{v}'$ is a vector in $\text{span}(S)$ but not in $\text{span}(S \setminus \{\mathbf{v}_i\})$ for some integer i between 1 and n , then $\text{span}(S) = \text{span}((S \setminus \{\mathbf{v}_i\}) \cup \{\mathbf{v}'\})$.*

Proof. Let $S' = (S \setminus \{\mathbf{v}_i\}) \cup \{\mathbf{v}'\}$. Since $\mathbf{v}' \in \text{span}(S)$ and $\mathbf{v}' \notin \text{span}(S \setminus \{\mathbf{v}_i\})$, there exists $c_1, \dots, c_n \in \mathbb{R}$ ($c_i \neq 0$) such that

$$\mathbf{v}' = c_1 \mathbf{v}_1 + \dots + c_n \mathbf{v}_n. \quad (2.4)$$

Hence, (2.4) can be modified as

$$\mathbf{v}_i = \frac{1}{c_i} \mathbf{v}' - \frac{c_1}{c_i} \mathbf{v}_1 - \dots - \frac{c_{i-1}}{c_i} \mathbf{v}_{i-1} - \frac{c_{i+1}}{c_i} \mathbf{v}_{i+1} - \dots - \frac{c_n}{c_i} \mathbf{v}_n. \quad (2.5)$$

If $\mathbf{u}$ is a vector in $\text{span}(S')$, then there exists $d_1, \dots, d_n \in \mathbb{R}$ such that

$$\mathbf{u} = d_1 \mathbf{v}_1 + \dots + d_i \mathbf{v}' + \dots + d_n \mathbf{v}_n. \quad (2.6)$$

Substitute (2.4) into (2.6) to express $\mathbf{u}$ as a linear combination of S , which implies that $\mathbf{u} \in \text{span}(S)$. Hence, $\text{span}(S') \subseteq \text{span}(S)$.

This time, if $\mathbf{u}$ is a vector in $\text{span}(S)$, then there exists $d'_1, \dots, d'_n \in \mathbb{R}$ such that

$$\mathbf{u} = d'_1 \mathbf{v}_1 + \dots + d'_n \mathbf{v}_n. \quad (2.7)$$

Substitute (2.5) into (2.7) to express $\mathbf{u}$ as a linear combination of S' , which implies that $\mathbf{u} \in \text{span}(S')$. Hence, $\text{span}(S) \subseteq \text{span}(S')$ and therefore, the theorem is proven. ■

The above lemma means that exchanging $\mathbf{v}_i$ with $\mathbf{v}'$ in S does not affect the span of S . We can use this fact to exchange the vectors in S whilst preserving its span. It can also be used to prove the following theorem.

Theorem 2.42 (Steinitz Exchange Lemma). *Let S, T be sets of vectors in vector space V where S is linearly independent and T spans V . Then:*

- (a) $m \leq n$.
- (b) *There exists $T' \subseteq T$ of cardinality $n - m$ such that $T' \cup S$ spans V .*

Proof. Let $S = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ and $T = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. Our idea is to use Lemma 2.41 to repeat a certain process which will end up replacing elements in T with elements in S .

T is a set of vectors such that $\text{span}(T) = V$. Hence, there exists some integer i between 1 and n such that $\mathbf{u}_1 \in \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_i\}$ and $\mathbf{u}_1 \notin \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_{i-1}\}$. By Lemma 2.41,

$$\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_i\} = \text{span}\{\mathbf{u}_1, \mathbf{v}_1, \dots, \mathbf{v}_{i-1}\}$$

and

$$\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\} = \text{span}\{\mathbf{u}_1, \mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n\}.$$

Thus, if $T_1 = \{\mathbf{u}_1, \mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n\}$, then T_1 is still a spanning set of V . Lastly, relabel elements $\mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n$ so that $T_1 = \{\mathbf{u}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, for the sake of convenience.

Now we repeat the above process. This is possible because $\mathbf{u}_k$ is always in $\text{span}(T_{k-1}) = \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_{k-1}, \mathbf{v}_k, \dots, \mathbf{v}_n\}$, and is not in $\text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_{k-1}\}$, which implies that there exists integer i between 1 and n such that exchanging $\mathbf{v}_i$ with $\mathbf{u}_k$ does not affect the span.

Hence, the process can be repeated until S is depleted and we end up with $T_n = \{\mathbf{u}_1, \dots, \mathbf{u}_m, \mathbf{v}_{m+1}, \dots, \mathbf{v}_n\}$. Therefore, $m \leq n$ and there exists a subset of T with cardinality $n - m$, $T' = \{\mathbf{v}_{m+1}, \dots, \mathbf{v}_n\}$, such that $T' \cup S$ spans V . ■

Theorem 2.42 (a) implies that the cardinality of linearly independent sets is always smaller or equal to the cardinality of spanning sets. Also, whilst proving the, we constructed T' , a subset of T , that allows $T' \cup S$ to span V .

This theorem will be directly used as a lemma to show that all bases of a finite-dimensional vector space have equal cardinality. But before we do that, we need guarantee that all finite-dimensional vector spaces have a basis. Hence, we look at the following theorem.

Theorem 2.43. *Suppose V is a finite-dimensional vector space. Then:*

- (a) *If S is a linearly independent set of vectors in V , then S can be extended to a basis for V by adding some vectors to S .*
- (b) *If T is a set of vectors in V that spans V , then T can be reduced to a basis for V by removing some vectors from T .*

Proof. (a) Suppose $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a linearly independent set of vectors in V . If S spans V , then S is already a basis for V .

Otherwise, if S does not span V , then there exists $\mathbf{v}' \in V \setminus \text{span}(S)$. Let $S_1 = S \cup \{\mathbf{v}'_1\}$. Then S_1 remains linearly independent, and if $\text{span}(S_1) = V$, then S_1 is a basis for V . If S_1 is yet to span V , then repeat the process and add more appropriate vectors to S_1 until it becomes a basis for V . Hence, S can be extended to a basis.

(b) Suppose $T = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a spanning set of V . If T is linearly independent, it is already a basis for V . Otherwise, if T is linearly dependent, then by Theorem 2.31, there exists $\mathbf{v}_i \in \text{span}(S)$ such that $\text{span}(S) = \text{span}(S \setminus \{\mathbf{v}_i\})$. Hence, we can delete $\mathbf{v}_i$ from T while preserving its span. We may repeat this process until T becomes linearly independent, since Theorem 2.31 works for all linearly dependent sets. Therefore, T can be reduced to a basis. ■

Theorem 2.43 primarily implies that we can construct a basis out of linearly independent or spanning sets. It turns out to be useful when solving problems that require us to find the basis of a given vector space. But the theorem also implies the following important corollary:

Corollary 2.44. *All finite-dimensional vector spaces have a basis.*

Proof. By definition, a finite-dimensional vector space V can be spanned by a finite set. Since Theorem 2.43 suggests that the spanning set can be reduced to a basis, V always has a basis. ■

Now that we are given Corollary 2.44, we can assume that a basis always exists and use Theorem 2.42 to prove the following theorem.

Theorem 2.45. *All bases of finite-dimensional vector space V have the same cardinality.*

Proof. By Corollary 2.44, V always has a basis. Hence, let $\mathcal{B}_1, \mathcal{B}_2$ be bases for V and suppose $|\mathcal{B}_1| = n$. By Theorem 2.42, a linearly independent set has at most n vectors, and a spanning set of V has at least n vectors. Therefore, basis $\mathcal{B}_2$, which is linearly independent and a spanning set of V , has a cardinality of exactly n . Therefore, all bases of V have the same cardinality. ■

We have finally proven the second fundamental theorem. Although the basis of finite-dimensional vector space V may not be unique, they all have the same cardinality. Since this is a common characteristic of all bases for V , it may be seen as a unique characteristic of V that determines its structure. We call this the dimension of a vector space, and give the following definition.

Definition 2.46. *If finite-dimensional vector space V has bases of n vectors, then we say that the **dimension** of V is equal to n , and denote $\dim(V) = n$. If $\dim(V) = n$, we call V an n -dimensional vector space.*

Remark. *If V is the zero vector space $\{\mathbf{0}\}$, then we conventionally define the dimension of V as 0. Hence, $\dim(V) = 0$.*

The dimension of a vector space V is the same as the maximal number of vectors in a linearly independent set and the minimal number of vectors in a spanning set of V . Hence, the basis is also a maximal linearly independent set and a minimal spanning set of V .

Example 2.47. Consider the following vector spaces:

- The vector space $\mathbb{R}^n$ has a dimension of n , since $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$ with cardinality n . Therefore, $\dim(\mathbb{R}^n) = n$.
- The vector space $\mathbb{P}_n$ has a dimension of $n + 1$, since $\{1, t, \dots, t^n\}$ is a basis for $\mathbb{P}_n$ with cardinality $n + 1$. Therefore, $\dim(\mathbb{P}_n) = n + 1$.

Recall that in Definition 2.38, we defined the coordinate vector. If V is an n -dimensional vector space, then all vectors in V have a coordinate vector from $\mathbb{R}^n$. Hence, any abstract vectors from an n -dimensional vector space can be expressed as a vector from $\mathbb{R}^n$, a more intuitive vector space.

The next theorem provides a loose criterion for a set of vectors to be a basis for some finite-dimensional vector space.

Theorem 2.48. Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a set of n vectors in n -dimensional vector space V . Then, S spans V if and only if S is linearly independent.

Proof. ($\Rightarrow$) Suppose S spans V and is linearly dependent. Then, by Theorem 2.31, we can remove a vector from S whilst preserving its span. But this is a contradiction since a set of less than n vectors can not span V by Theorem 2.42. ($\Leftarrow$) Suppose S is linearly independent but does not span V . Then, by Theorem 2.43 (a), we can add vectors to S so that S becomes a basis for V . But this is a contradiction since a set of more than n vectors can not be linearly independent by Theorem 2.42. ■

Since S is a spanning set if and only if S is linearly independent, according to the above theorem, S is a basis for V as long as we check that S is either a spanning set or a linearly independent set. Hence, if we have that the cardinality of the given set is equal to the dimension of V , we only have to check either of the two criterion to show that the given set is a basis for V .

Dimension of Subspaces

By intuition, we may suspect that the dimension of a subspace W of V is either equal to or smaller than the dimension of V , since W consists a part of V . The following theorem verifies our intuition in a rigorous manner.

Theorem 2.49. Suppose V is a finite-dimensional vector space and W is a subspace of V . Then:

- W is finite-dimensional.
- $\dim(W) \leq \dim(V)$.

(c) $W = V$ if and only if $\dim(W) = \dim(V)$.

Proof. (a) Suppose W is not finite-dimensional. Then, an infinite set of vectors in W may be linearly independent, and since there is no finite spanning set for W , its basis is also an infinite set. This implies that V is infinite-dimensional, which is a contradiction. Hence, W has to be finite-dimensional.

(b) By (a), W is finite-dimensional, so there exists a basis $\mathcal{B}$ for W . If $\mathcal{B}$ is also a basis for V , then $\dim(W) = \dim(V)$. But if $\mathcal{B}$ is not a basis for V , we can add some vectors to $\mathcal{B}$ so that $\mathcal{B}$ becomes a basis for V , due to Theorem 2.43 (a). Then, $\dim(W) < \dim(V)$.

(c) ($\Rightarrow$) If $W = V$, then it is obvious that $\dim(W) = \dim(V)$.

($\Leftarrow$) Suppose $\dim(W) = \dim(V)$ and $\mathcal{B}$ is a basis for W . If $\mathcal{B}$ is not a basis for V , then $\mathcal{B}$ can be extended to a basis for V due to Theorem 2.43 (a). But this contradicts Theorem 2.42, hence $\mathcal{B}$ is also a basis for V . Therefore, $W = V$. ■

(b) and (c) combined together suggests that W and V have the same dimension only when $W = V$, and if $W \subsetneq V$, then $\dim(W)$ is always smaller than $\dim(V)$.

Next, we look at the dimension of the sum of subspaces. The following theorem provides a specific formula to calculate the dimension of the sum.

Theorem 2.50. *If W_1, W_2 are subspaces of a finite-dimensional vector space V , then*

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2). \quad (2.8)$$

Proof. Suppose $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ is a basis for $W_1 \cap W_2$. By Theorem 2.43 (a), we can extend $\mathcal{B}$ so that it becomes the basis for W_1 and W_2 . Hence, let $\mathcal{B}_1 = \mathcal{B} \cup \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a basis for W_1 and $\mathcal{B}_2 = \mathcal{B} \cup \{\mathbf{w}_1, \dots, \mathbf{w}_q\}$ be a basis for W_2 . Claim that $\mathcal{B}' = \mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for $W_1 + W_2$.

First, show that $\mathcal{B}'$ spans $W_1 + W_2$. Suppose $\mathbf{x}$ is a vector in $W_1 + W_2$ such that $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ where $\mathbf{x}_1 \in W_1$ and $\mathbf{x}_2 \in W_2$. Then, there exists $c_1, \dots, c_{m+p}, d_1, \dots, d_{m+q} \in \mathbb{R}$ such that

$$\mathbf{x}_1 = c_1 \mathbf{u}_1 + \dots + c_m \mathbf{u}_m + c_{m+1} \mathbf{v}_1 + \dots + c_{m+p} \mathbf{v}_p$$

and

$$\mathbf{x}_2 = d_1 \mathbf{u}_1 + \dots + d_m \mathbf{u}_m + d_{m+1} \mathbf{w}_1 + \dots + d_{m+q} \mathbf{w}_q.$$

Hence, $\mathbf{x}$ is in $\text{span } \mathcal{B}'$ and $\text{span } \mathcal{B}' = W_1 + W_2$.

Next, show that $\mathcal{B}'$ is linearly independent. Suppose $a_1, \dots, a_m, b_1, \dots, b_p, c_1, \dots, c_q \in \mathbb{R}$ are scalars such that

$$a_1 \mathbf{u}_1 + \dots + a_m \mathbf{u}_m + b_1 \mathbf{v}_1 + \dots + b_p \mathbf{v}_p + c_1 \mathbf{w}_1 + \dots + c_q \mathbf{w}_q = \mathbf{0}. \quad (2.9)$$

Then, we can adjust the equation as

$$a_1 \mathbf{u}_1 + \dots + a_m \mathbf{u}_m + b_1 \mathbf{v}_1 + \dots + b_p \mathbf{v}_p = -(c_1 \mathbf{w}_1 + \dots + c_q \mathbf{w}_q), \quad (2.10)$$

where the left hand side is a vector in W_1 and the right hand side is a vector in W_2 . Hence, the vector above is a vector in $W_1 \cap W_2$. Since $\mathcal{B}$ is a basis for $W_1 \cap W_2$,

we can express the right hand side of (2.10) as

$$-(c_1 \mathbf{w}_1 + \cdots + c_q \mathbf{w}_q) = d_1 \mathbf{u}_1 + \cdots + d_m \mathbf{u}_m$$

where $d_1, \dots, d_m \in \mathbb{R}$, and adjust it again as

$$c_1 \mathbf{w}_1 + \cdots + c_q \mathbf{w}_q + d_1 \mathbf{u}_1 + \cdots + d_m \mathbf{u}_m = \mathbf{0}.$$

Recall that $\mathcal{B}_2$ is a basis for W_2 , which gives that $\mathcal{B}_2$ is linearly independent and hence, $c_1 = \cdots = c_q = d_1 = \cdots = d_m = 0$. Substitute $c_1 = \cdots = c_q = 0$ into (2.9) to get

$$a_1 \mathbf{u}_1 + \cdots + a_m \mathbf{u}_m + b_1 \mathbf{v}_1 + \cdots + b_p \mathbf{v}_p = \mathbf{0},$$

which again leads to $a_1 = \cdots = a_m = b_1 = \cdots = b_p = 0$, since $\mathcal{B}_1$ is also linearly independent. Thus, $\mathcal{B}'$ is linearly independent. Therefore, $\mathcal{B}'$ is a basis for $W_1 + W_2$, which implies that $\dim(W_1 + W_2) = m + p + q$. Finally, we derive the following equation:

$$\begin{aligned} \dim(W_1 + W_2) &= m + p + q = (m + p) + (m + q) - m \\ &= \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2). \end{aligned}$$

■

Example 2.51. If $\dim(W_1) = 2$, $\dim(W_2) = 3$, and $\dim(W_1 + W_2) = 4$, then using Theorem 2.50, we can find that $\dim(W_1 \cap W_2) = 1$.

We may also consider the dimension of the direct sum of subspaces, which can be easily derived from the above theorem.

Corollary 2.52. Let W_1, W_2 be subspaces of a finite-dimensional vector space V . Then, $W_1 + W_2$ is a direct sum if and only if

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2). \quad (2.11)$$

Proof. ($\Rightarrow$) If $W_1 + W_2$ is a direct sum, then $W_1 \cap W_2 = \{\mathbf{0}\}$. Therefore,

$$\dim(W_1 \cap W_2) = 0,$$

and substituting this to (2.8) gives (2.11).

($\Leftarrow$) If $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2)$, then $\dim(W_1 \cap W_2) = 0$. Hence, $W_1 \cap W_2 = \{\mathbf{0}\}$, which implies that $W_1 + W_2$ is a direct sum. ■

Hence we can rewrite (2.11) as

$$\dim(W_1 \oplus W_2) = \dim(W_1) + \dim(W_2).$$

Example 2.53. In Theorem 2.50, we found the formula

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

Recall that in Theorem 2.19, we proved that if W_1, W_2 are subspaces of vector space V , then $W_1 + W_2$ is a subspace of V that contains W_1 and W_2 . Hence, W_1 and W_2 are subspaces of $W_1 + W_2$. So by Theorem 2.49 (b),

$$\dim(W_1) \leq \dim(W_1 + W_2) \quad \text{and} \quad \dim(W_2) \leq \dim(W_1 + W_2)$$

must be satisfied, and we can check this fact using Theorem 2.50. Since $W_1 \cup W_2$ is a subspace of $W_1 + W_2$ by Theorem 2.17,

$$\dim(W_1 \cup W_2) \leq \dim(W_1) \quad \text{and} \quad \dim(W_1 \cup W_2) \leq \dim(W_2)$$

which leads to

$$\dim(W_1) \leq \dim(W_1) + \dim(W_2) - \dim(W_1 \cup W_2) = \dim(W_1 + W_2)$$

$$\dim(W_2) \leq \dim(W_1) + \dim(W_2) - \dim(W_1 \cup W_2) = \dim(W_1 + W_2).$$

3 | Linear Systems

Historically, linear algebra started from the question of linear systems, a system of linear equations. How can we find its solutions? Does the solution always exist? Is it unique? These questions form a core part of linear algebra, and will be properly dealt in this chapter.

But before we deal with linear systems, we introduce an important tool used to represent linear systems: the matrix. Matrices are objects with its own unique operations that make it easier to express linear systems. Not only do they make notations more compact, but they also simplify the problems we will be dealing with. We shall look at the beauty of matrices under the context of linear systems.

3.1 Matrices and its Operations

Matrices are widely used throughout mathematics, physics, and computer science. It stores numerical data in an organized structure so that we can express complicated equations in a simple manner and minimize unnecessary calculations.

In linear algebra, matrices are used to express vectors, linear systems, and linear maps. Its unique algebraic properties lead us to a fundamental theorem that relates all core topics of linear algebra. Hence, matrices serve to be a crucial tool in linear algebra.

In this section, we look at what matrices are and how its operations are defined. We will also see why matrices arise naturally in the context of linear algebra.

Definition of a Matrix

Definition 3.1. A **matrix**¹ is a rectangular array of numbers. Each number in a given matrix is called the **entry** of the matrix.

Example 3.2. The following are examples of some matrices.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} -\sqrt{2} & \frac{1}{2} & 0 \\ \pi & e & 32 \end{bmatrix}, \quad \begin{bmatrix} 1+3i & 2-i & -1 \\ 0 & \sqrt{-1} & 0 \\ 1 & 2i & 3-2i \end{bmatrix}, \quad [32]$$

¹Although the entry of a matrix can be any number, including complex numbers, we only deal with matrices of real entries in this book.

We describe the **size** of a matrix by its number of rows (horizontal lines) and columns (vertical lines). For example, if a given matrix is consisted of m rows and n columns, we say that the matrix has a size of $m \times n$. Hence, a matrix of size $m \times n$ can be expressed as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

A can also be denoted as $[a_{ij}]_{m \times n}$ or $[a_{ij}]$, expressing that A is a matrix of entries a_{ij} , and entry a_{ij} can also be denoted as $(A)_{ij}$, expressing that a_{ij} is an entry of A .

Matrices can be used to express vectors in $\mathbb{R}^n$. For example, if $\mathbf{u} = (u_1, \dots, u_n)$, a vector in $\mathbb{R}^n$, then we can express $\mathbf{u}$ as either

$$\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \quad \text{or} \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}. \quad (3.1)$$

The matrix on the left is called a **row matrix** or a **row vector**, and the matrix on the right is called a **column matrix** or a **column vector**. Either way, the matrix fully represents the vector $\mathbb{R}^n$ in replacement of the typical n -tuple representation.

A general matrix of size $m \times n$ may also be interpreted as either a list of m row vectors or a list of n column vectors. Hence, if A is an $m \times n$ matrix, then the i th **row vector** of A , denoted as $\mathbf{r}_i(A)$, is defined as

$$\mathbf{r}_i(A) = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix},$$

whereas the j th **column vector** of A , denoted as $\mathbf{c}_j(A)$, is defined as

$$\mathbf{c}_j(A) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}.$$

Using row vectors or column vectors, we can express A as

$$A = \begin{bmatrix} \mathbf{r}_1(A) \\ \mathbf{r}_2(A) \\ \vdots \\ \mathbf{r}_m(A) \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1(A) & \mathbf{c}_2(A) & \cdots & \mathbf{c}_n(A) \end{bmatrix}.$$

As the above expression suggests, a given matrix may be divided into **submatri-**

ces. The following example shows a more general example of this.

Example 3.3. Suppose

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 2 & 0 & 1 & 3 \\ -2 & 4 & 2 & 0 \end{bmatrix}$$

is given. If

$$A_{11} = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} -2 & 4 & 2 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0 \end{bmatrix},$$

then we can express A as

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 2 & 0 & 1 & 3 \\ -2 & 4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Definition 3.4. Matrix A with n rows and n columns is called a **square matrix** of order n . We say that entries $(A)_{11}, (A)_{22}, \dots, (A)_{nn}$ are on the **main diagonal** of the square matrix.

Example 3.5. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

Then, A can be called a square matrix of order 4, since it is a 4×4 matrix. The main diagonal of A includes the entries 1, 6, 11, 16.

Matrix Operations

Now we look at the operations of matrices. Consider not only the result of the operation, but the prerequisite condition for the operations to be done.

Definition 3.6. Two matrices A and B are called **equal** if they have the same size and

$$(A)_{ij} = (B)_{ij} \quad \text{for all } i, j.$$

Example 3.7. Consider the matrices

$$A = \begin{bmatrix} 1 & 3 & x \\ y & 4 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

Comparing matrices A and B , we see that they have the same size. For all their corresponding entries to be equal, x must be 5 and y must be 2. Hence if $x = 5$ and $y = 2$, then $A = B$.

Comparing matrices A and C , we see that they have different sizes. Thus, they can never be equal. Same applies to B and C .

Definition 3.8. If A and B are $m \times n$ matrices, then the **sum** $A + B$ is an $m \times n$ matrix where

$$(A + B)_{ij} = (A)_{ij} + (B)_{ij} \quad \text{for all } i, j.$$

Definition 3.9. If A and B are $m \times n$ matrices, then the **difference** $A - B$ is an $m \times n$ matrix where

$$(A - B)_{ij} = (A)_{ij} - (B)_{ij} \quad \text{for all } i, j.$$

Remark. Notice that the sum and difference of matrices can not be defined when the two given matrices are of different sizes.

The sum and difference of matrices is similar to the sense of sum and difference of vectors in $\mathbb{R}^n$. This is because matrices serve the role of expressing vectors in $\mathbb{R}^n$, as we have seen in (3.1). Same applies to the scalar multiplication of matrices, defined below.

Definition 3.10. If A is an $m \times n$ matrix and c is any scalar in $\mathbb{R}$, then the **scalar multiple** cA is an $m \times n$ matrix where

$$(cA)_{ij} = c \cdot (A)_{ij} \quad \text{for all } i, j.$$

Example 3.11. Given the matrices

$$A = \begin{bmatrix} 3 & -1 & 1 & 0 \\ 2 & 5 & -2 & 9 \\ 4 & -4 & 7 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 2 & 3 & 4 \\ -1 & 3 & -4 & -2 \\ 0 & 0 & 3 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$

find (a) $A + B$ (b) $A - B$ (c) $3C$.

Solution. (a) By Definition 3.8,

$$A + B = \begin{bmatrix} 9 & 1 & 4 & 4 \\ 1 & 8 & -6 & 7 \\ 4 & -4 & 10 & 7 \end{bmatrix}.$$

(b) By Definition 3.9,

$$A - B = \begin{bmatrix} -3 & -3 & -2 & -4 \\ 3 & 2 & 2 & 11 \\ 4 & -4 & 4 & -3 \end{bmatrix}.$$

(c) By Definition 3.10,

$$3C = \begin{bmatrix} 3 \cdot 1 & 3 \cdot 2 \\ 3 \cdot 3 & 3 \cdot 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}.$$

Notice that $A + C$ and $B + C$ can not be defined, since their sizes are different.

Next, we define the product between two matrices.

Definition 3.12. If A is a $m \times r$ matrix and B is a $r \times n$ matrix, then the **product** AB is an $m \times n$ matrix where

$$(AB)_{ij} = (A)_{i1}(B)_{1j} + \cdots + (A)_{ir}(B)_{rj} = \sum_{k=1}^r (A)_{ik}(B)_{kj} \quad \text{for all } i, j.$$

Remark. Notice that for the product AB to be defined, the sizes of A and B have to be of the form $m \times r$ and $r \times n$, respectively. This is because for each element of AB , we have to operate with the row vector of A and the column vector of B .

Suppose that A and B are given as

$$A = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix}, \quad B = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n]. \quad (3.2)$$

Then, the product AB can be written as

$$AB = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \mathbf{a}_1 \cdot \mathbf{b}_2 & \cdots & \mathbf{a}_1 \cdot \mathbf{b}_n \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \mathbf{a}_2 \cdot \mathbf{b}_2 & \cdots & \mathbf{a}_2 \cdot \mathbf{b}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_m \cdot \mathbf{b}_1 & \mathbf{a}_m \cdot \mathbf{b}_2 & \cdots & \mathbf{a}_m \cdot \mathbf{b}_n \end{bmatrix}$$

using the Euclidean inner product. Note that the length of the row vectors of A and the column vectors of B must be the same for the inner product to be defined.

The reason why the matrix multiplication is defined this way is because it simplifies the matrix notation for linear systems and linear maps. We will look further into this later.

Example 3.13. If

$$A = \begin{bmatrix} -1 & 4 & 2 \\ 0 & 6 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 3 \\ 2 & -1 \\ 4 & 3 \end{bmatrix},$$

then

$$AB = \begin{bmatrix} (-1) \cdot 0 + 4 \cdot 2 + 2 \cdot 4 & (-1) \cdot 3 + 4 \cdot (-1) + 2 \cdot 3 \\ 0 \cdot 0 + 6 \cdot 2 + (-3) \cdot 4 & 0 \cdot 3 + 6 \cdot (-1) + (-3) \cdot 3 \end{bmatrix} = \begin{bmatrix} 16 & -1 \\ 0 & -15 \end{bmatrix}.$$

Given (3.2), we may also express the product AB as

$$AB = A[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n] = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_n]$$

or

$$AB = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix} B = \begin{bmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_m B \end{bmatrix}.$$

This gives that

$$\mathbf{c}_j(AB) = A\mathbf{c}_j(B)$$

and

$$\mathbf{r}_i(AB) = \mathbf{r}_i(A)B.$$

These properties are useful when handling the product of general matrices. It simplifies the formulas we have to handle.

The following operation is an operation unique to matrices. It flips the given matrix in respect to its main diagonal.

Definition 3.14. If A is an $m \times n$ matrix, then the **transpose** of A , denoted as A^T , is a $n \times m$ matrix such that

$$(A^T)_{ij} = (A)_{ji} \quad \text{for all } i, j.$$

Example 3.15. Given the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix},$$

find its transposes.

Solution.

$$A^T = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & 1 \\ 4 & 3 \\ 6 & 5 \end{bmatrix}.$$

Properties of Matrix Operations

So far, we have observed how the addition, subtraction, scalar multiple, product, and transpose of matrices is defined. In this section, we look at the basic properties of these operations.

Theorem 3.16. Suppose that A, B, C are matrices and k, l are scalars in $\mathbb{R}$ given so that the operations below can be defined. Then:

- (a) $A + B = B + A$.
- (b) $A + (B + C) = (A + B) + C$.
- (c) $A(BC) = (AB)C$.

- (d) $A(B + C) = AB + AC$.
- (e) $(A + B)C = AC + BC$.
- (f) $k(lA) = (kl)A$.
- (g) $(k + l)A = kA + lA$.
- (h) $k(A + B) = kA + kB$.
- (i) $A(kB) = k(AB) = (kA)B$.
- (j) $(A^T)^T = A$.
- (k) $(A + B)^T = A^T + B^T$.
- (l) $(AB)^T = B^T A^T$.
- (m) $(kA)^T = kA^T$.

Recall Definition 3.6, which provides the conditions for when two given matrices are equivalent. We will use this for our proof to properties (a) to (d) below. Proofs for the rest of the properties are left as **Exercise**.

Proof. (a) First, check that $A + B$ and $B + A$ are matrices of the same size. Next, check that each element of both matrices are equivalent:

$$(A + B)_{ij} = (A)_{ij} + (B)_{ij} = (B)_{ij} + (A)_{ij} = (B + A)_{ij}.$$

Thus, $A + B = B + A$.

(b) Same as (a), we first check that $A + (B + C)$ and $(A + B) + C$ are matrices of the same size. Then, we check that each element of both matrices are equivalent:

$$\begin{aligned} (A + (B + C))_{ij} &= (A)_{ij} + ((B)_{ij} + (C)_{ij}) \\ &= ((A)_{ij} + (B)_{ij}) + (C)_{ij} = ((A + B) + C)_{ij}. \end{aligned}$$

Hence, $A + (B + C) = (A + B) + C$.

(c) Suppose A is a $m \times r$ matrix, B is a $r \times s$ matrix, and C is a $s \times n$ matrix. Then, $A(BC)$ and $(AB)C$ are both matrices of size $m \times n$.

Now we shall compare the entries of each matrices. First, the ij th entry of $A(BC)$ is

$$\begin{aligned} (A(BC))_{ij} &= \sum_{k=1}^r (A)_{ik} (BC)_{kj} \\ &= \sum_{k=1}^r (A)_{ik} \left(\sum_{l=1}^s (B)_{kl} (C)_{lj} \right) \\ &= \sum_{k=1}^r \sum_{l=1}^s (A)_{ik} (B)_{kl} (C)_{lj}. \end{aligned}$$

Similarly, the ij th entry of $(AB)C$ is

$$\begin{aligned} ((AB)C)_{ij} &= \sum_{l=1}^s (AB)_{il} (C)_{lj} \\ &= \sum_{l=1}^s \left(\sum_{k=1}^r (A)_{ik} (B)_{kl} \right) (C)_{lj} \\ &= \sum_{l=1}^s \sum_{k=1}^r (A)_{ik} (B)_{kl} (C)_{lj}. \end{aligned}$$

Hence, we can check that $(A(BC))_{ij} = ((AB)C)_{ij}$ for all i, j and therefore, (c) is proven.

(d) If A is a $m \times r$ matrix and B, C are $r \times n$ matrices, then both $A + (B + C)$ and $(A + B) + C$ are $m \times n$ matrices. Also,

$$\begin{aligned} (A(B + C))_{ij} &= (A)_{i1}(B + C)_{1j} + (A)_{i2}(B + C)_{2j} + \cdots + (A)_{ir}(B + C)_{rj} \\ &= a_{i1}(b_{1j} + c_{1j}) + a_{i2}(b_{2j} + c_{2j}) + \cdots + a_{ir}(b_{rj} + c_{rj}) \\ &= (a_{i1}b_{1j} + \cdots + a_{ir}b_{rj}) + (a_{i1}c_{1j} + \cdots + a_{ir}c_{rj}) \\ &= (AB)_{ij} + (AC)_{ij} = (AB + AC)_{ij}. \end{aligned}$$

Therefore, $A(B + C) = AB + AC$. ■

Example 3.17. *Given*

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -2 \end{bmatrix},$$

we find that

$$A(BC) = \begin{bmatrix} 1 & 0 & -2 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 6 & -6 \\ -2 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 14 & -14 \end{bmatrix}$$

and

$$(AB)C = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \begin{bmatrix} 2 & -2 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 14 & -14 \end{bmatrix}.$$

Thus, we can check that $A(BC) = (AB)C$.

It is necessary that we get familiar with the properties given in Theorem 3.16, since they are the most basic algebraic properties of matrices. Also, be cautious of mindlessly justifying any algebraic properties that is, in fact, not true. For example, $AB \neq BA$ in general, as we shall examine in the following example.

Example 3.18. Suppose

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Then,

$$AB = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 7 & 1 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ 3 & 3 \end{bmatrix}.$$

Thus, we see that $AB \neq BA$.

As the above example suggests, not all common properties may be true in general. So before applying an arithmetic property, check whether it is a generalized property for all matrices.

Example 3.19. Consider the set of all $m \times n$ matrices with real entries, denoted as $M_{m \times n}(\mathbb{R})$. Under Definition 3.8 and 3.10, we can check that $M_{m \times n}$ satisfies all vector space axioms in Definition 2.1. Hence, $M_{m \times n}$ is a vector space and all matrices are vectors. (This is an obvious result since the motivation for matrix addition and scalar multiplication came from the operations of vectors in $\mathbb{R}^n$.)

Also, let M_{ab} be an $m \times n$ matrix such that

$$(M_{ab})_{ij} = \begin{cases} 1 & \text{if } (i, j) = (a, b) \\ 0 & \text{else.} \end{cases}$$

Then, $\mathcal{B} = \{M_{11}, M_{12}, \dots, M_{mn}\}$ forms a basis for $M_{m \times n}(\mathbb{R})$. We call $\mathcal{B}$ the standard basis for $M_{m \times n}(\mathbb{R})$, and since the cardinality of $\mathcal{B}$ is mn ,

$$\dim(M_{m \times n}(\mathbb{R})) = mn.$$

Now we look at the additive identity and the additive inverse of matrices.

Definition 3.20. An $m \times n$ matrix whose entries are all zero is called a **zero matrix** of size $m \times n$, and we denote it as $\mathbf{0}_{m \times n}$ or simply $\mathbf{0}$. Hence,

$$\mathbf{0}_{m \times n} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{m \times n}.$$

Theorem 3.21. Suppose A is an $m \times n$ matrix and $\mathbf{0}$ the zero matrix of size $m \times n$. Then:

(a) $A + \mathbf{0} = \mathbf{0} + A = A.$

(b) $A + (-1)A = \mathbf{0}.$

(c) $cA = \mathbf{0}$ if and only if $c = 0$ or $A = \mathbf{0}$ (where c is a scalar).

Proof. Exercise. ■

Theorem 3.21 (a) implies that $\mathbf{0}$ is the identity element for matrix addition, and (b) implies that $(-1)A$ is the inverse element of A for matrix addition. We may denote this as $-A = (-1)A$.

Example 3.22. *Given*

$$A = \begin{bmatrix} 0 & 3 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix},$$

we can check that although $AB = \mathbf{0}$, A and B are not zero matrices.

Next, we look the multiplicative identity and the multiplicative inverse of matrices.

Definition 3.23. *An $n \times n$ matrix is called an **identity matrix** and denoted as I_n if*

$$(I_n)_{ij} = \delta_{ij} \quad \text{for all } i, j,$$

where δ_{ij} is the Kronecker delta.

Theorem 3.24. *If A is an $m \times n$ matrix, then*

$$I_m A = A \quad \text{and} \quad A I_n = A.$$

Proof. $I_m A$ and A have same sizes of $m \times n$. Comparing their entries,

$$\begin{aligned} (I_m A)_{ij} &= \mathbf{r}_i(I_m) \cdot \mathbf{c}_j(A) \\ &= \sum_{k=1}^m \delta_{ik} \cdot (A)_{kj} \\ &= \delta_{ii} \cdot (A)_{ij} = (A)_{ij}. \end{aligned}$$

Therefore, $I_m A = A$.

$A I_n$ and A also have same sizes of $m \times n$. Again, comparing their entries,

$$\begin{aligned} (A I_n)_{ij} &= \mathbf{r}_i(A) \cdot \mathbf{c}_j(I_n) \\ &= \sum_{k=1}^n (A)_{ik} \cdot \delta_{kj} \\ &= (A)_{ij} \cdot \delta_{jj} = (A)_{ij}. \end{aligned}$$

Therefore, $A I_n = A$. ■

Hence, I_n is the multiplicative identity of matrices. When we deal with multiplicative inverses, we have to consider whether the matrix is multiplied on the left or the right of the given matrix since matrix multiplication is not commutative, as we saw in Example 3.18. Thus, we start with the following definition.

Definition 3.25. For $m \times n$ matrix A , an $n \times m$ matrix B is called a **left inverse** of A if $BA = I_n$, and an $n \times m$ matrix C is called a **right inverse** of A if $AC = I_m$.

So far, we do not have a guarantee that some $m \times n$ matrix A will always have a left inverse or a right inverse. Nor do we have guarantee that the left inverse and the right inverse of A are the same. But it turns out that square matrices are quite special in this sense. The following theorem shows why.

Theorem 3.26. If A is an $n \times n$ matrix with left inverse B and right inverse C , then $B = C$.

Proof. Since

$$BA = I_n \quad \text{and} \quad AC = I_n,$$

we have

$$B = BI_n = B(AC) = (BA)C = I_n C = C.$$

■

So if A is a square matrix and we are given the existence of its left inverse and right inverse, then the left inverse and the right inverse are the same. We do not have to distinct between a left inverse and a right inverse for square matrices.

Definition 3.27. Suppose A is a square matrix of size $n \times n$. If there exists a square matrix B of size $n \times n$ such that $BA = AB = I_n$, then A is said to be **invertible** and B is called an **inverse** of A , denoted as $B = A^{-1}$.

We still do not have a guarantee that all square matrices are invertible, so it requires showing the existence of a left inverse and a right inverse of the square matrix to claim that the square matrix is invertible.

Theorem 3.28. The inverse of square matrix A is unique.

Proof. Let B, C be the inverses of A and suppose $B \neq C$. Since B, C are inverses of A , we can say that A has a left inverse B and right inverse C . But since $B \neq C$, this contradicts Theorem 3.26. ■

The above theorem adds that the inverse matrix A^{-1} is unique. Hence, a square matrix has either one inverse matrix or none. We are yet to find out when a square matrix has an inverse and when it does not. Furthermore, we are yet to find out when a nonsquare matrix has either a left inverse or a right inverse. We shall discuss this later in this chapter.

Example 3.29. Let A be an invertible 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $ad - bc \neq 0$. Then, the inverse of A is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (3.3)$$

Equation (3.3) is the formula for the inverse matrix of a given 2×2 matrix when $ad - bc \neq 0$. We will look at methods to find the inverse matrix of an invertible $n \times n$ matrix later.

Theorem 3.30. *If A and B are invertible matrices, then AB is invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof. Since

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I,$$

AB is invertible and $B^{-1}A^{-1}$ is its inverse. ■

Theorem 3.31. *If $A_1, A_2, \dots, A_n$ are invertible matrices of size $n \times n$, then $A_1A_2 \cdots A_n$ is invertible and*

$$(A_1A_2 \cdots A_n)^{-1} = A_n^{-1}A_{n-1}^{-1} \cdots A_1^{-1}.$$

Proof. Use mathematical induction based on Theorem 3.30. Specifics are left for **Exercise**. ■

Theorem 3.32. *If A is invertible, then:*

- (a) A^{-1} is invertible and $(A^{-1})^{-1} = A$.
- (b) kA is invertible for any $k \neq 0$, and $(kA)^{-1} = k^{-1}A^{-1}$.

Proof. **Exercise**. ■

Theorem 3.33. *If A is invertible, then A^T is invertible and*

$$(A^T)^{-1} = (A^{-1})^T.$$

Proof. Since

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

$$(A^{-1})^TA^T = (AA^{-1})^T = I^T = I,$$

A^T is invertible and $(A^{-1})^T$ is its inverse. ■

Special Types of Matrices

We now look at some special matrices which will be useful in later chapters.

Definition 3.34. *Square matrix D where*

$$(D)_{ij} = 0 \quad \text{for all } i, j$$

is called a **diagonal matrix**.

In other words, diagonal matrices are matrices that have nonzero entries only on its main diagonal.

Theorem 3.35. Suppose D is a $n \times n$ diagonal matrix such that

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}.$$

Then, the power of D about a positive integer k is

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}.$$

Proof. **Exercise.** ■

Definition 3.36. Square matrix U where

$$(U)_{ij} = 0 \quad \text{for all } i > j$$

is called an **upper triangular matrix**. Square matrix L where

$$(L)_{ij} = 0 \quad \text{for all } i < j$$

is called a **lower triangular matrix**. If the given matrix is either a diagonal matrix, a lower triangular or an upper triangular matrix, then we call it a **triangular matrix**.

Theorem 3.37. The transpose of a lower triangular matrix is upper triangular, and vice versa.

Proof. **Exercise.** ■

Theorem 3.38. The inverse of an invertible lower triangular matrix is also lower triangular. The inverse of an invertible upper triangular matrix is also upper triangular.

Proof. **Exercise.** ■

Definition 3.39. A square matrix A is called **symmetric** if $A^T = A$. A square matrix A is called **skew-symmetric** if $A^T = -A$.

Theorem 3.40. All square matrices can be uniquely expressed as a sum of a symmetric matrix and a skew-symmetric matrix.

Proof. Suppose

$$A = A_1 + A_2, \quad (3.4)$$

where A_1 is symmetric and A_2 is skew-symmetric. Then,

$$A^T = (A_1)^T + (A_2)^T = A_1 - A_2. \quad (3.5)$$

By adding (3.4) to (3.5), we get

$$A_1 = \frac{A + A^T}{2}$$

and by subtracting (3.5) from (3.4), we get

$$A_2 = \frac{A - A^T}{2}.$$

From the formulas induced above, we can verify that A_1 is symmetric and A_2 is skew-symmetric. Also, we can check that A_1 and A_2 add up to A . Hence, all square matrices can be uniquely expressed as a sum of a symmetric matrix and a skew-symmetric matrix. ■

So far, we have looked at what matrices are and how their operations are defined. We also looked at their algebraic properties, including additive and multiplicative identities and inverses. Now we head to the main problem of this chapter, solving linear systems.

3.2 Linear Systems and its Solutions

A **linear equation** of n variables $x_1, x_2, \dots, x_n$ can be expressed in the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b, \quad (3.6)$$

where $a_1, a_2, \dots, a_n$ are constants that are not all zero, and b is also a constant. If $b = 0$ in (3.6), then it takes the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = 0$$

and is called a **homogeneous linear equation**.

A **system of linear equations** or a **linear system** is a finite set of linear equations. A linear system with m equations and n unknown variables $x_1, x_2, \dots, x_n$ looks like

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m. \end{cases} \quad (3.7)$$

The **solution** of a linear system with n unknowns is any n -tuple $(x_1, x_2, \dots, x_n)$

such that substituting each value to the unknown satisfies each equation of the system. The **solution set** of a given linear system is the set of all such possible solutions. Also, if a linear system has no solutions, then it is called **inconsistent**, and if a linear system has at least one solution, then it is called **consistent**.

The linear system in (3.7) can also be expressed in terms of matrices. Suppose we construct a matrix only out of the coefficients of the given equations as below:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}. \quad (3.8)$$

We call (3.8) the **augmented matrix** of the given linear system. Also, if

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

then we can denote the augmented matrix as $[A \mid \mathbf{b}]$ and express the given linear system as the matrix equation $A\mathbf{x} = \mathbf{b}$ since

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \mathbf{b}.$$

While the augmented matrix is just a simple way of representing a given linear system, the matrix equation $A\mathbf{x} = \mathbf{b}$ is a formal algebraic equation that has the same solution set as the given linear system. In other words, all column vectors $\mathbf{x}$ in $\mathbb{R}^n$ that is a solution of $A\mathbf{x} = \mathbf{b}$ is a solution for the corresponding linear system.

Now we look at some examples of linear systems and its solution sets.

Example 3.41. *Linear system*

$$\begin{cases} 2x - y = 1 \\ 3x + 2y = 12 \end{cases}$$

has exactly one solution $(x, y) = (2, 3)$.

Example 3.42. *Linear system*

$$\begin{cases} 2x + 2y = 4 \\ x + y = 1 \end{cases}$$

has no solution.

Example 3.43. *Linear system*

$$\begin{cases} 2x + 2y = 4 \\ x + y = 2 \end{cases}$$

has infinitely many solutions of the form $(x, y) = (2 - s, s)$, where s is any real number.

Let's consider the above examples in a geometric perspective. First of all, each equation in the linear systems is represented by a line in the $\mathbb{R}^2$ plane. For example, the equation $2x - y = 1$ is a line with slope 2 and x -intercept $(1/2, 0)$, y -intercept $(0, -1)$. Note that all points above this line is a solution of the equation $2x - y = 1$.

So for the linear system in Example 3.41, the solution would be the intersect between the lines $2x - y = 1$ and $3x + 2y = 12$, as the solution of a system must be a solution of all equations in the system. Since $2x - y = 1$ and $3x + 2y = 12$ are lines that are not parallel to each other, they will intersect at a single point. Hence, it has one solution, just as we saw in the example.

In Example 3.42, the two lines are parallel, but are not equivalent. Hence, they do not intersect each other, meaning that the system has no solution. But in Example 3.43, the two lines are parallel and equivalent, so all points above the line becomes a solution of the system.

Through this perspective, we arrive at a possible conclusion. Since two lines above a plane may either intersect at a single point, never intersect, or meet at infinitely many points, the linear system of two equations and two variables has a solution set of either one solution, no solution, or infinitely many solutions. In fact, this result applies to all linear systems, as we will see in the next theorem.

Theorem 3.44. *All linear systems have either one solution, no solution, or infinitely many solutions.*

Proof. Suppose A is an $m \times n$ matrix, $\mathbf{x}$ is a $n \times 1$ column vector, and $\mathbf{b}$ is a $m \times 1$ column vector such that $A\mathbf{x} = \mathbf{b}$ is the given linear system. To prove the theorem, claim that if $A\mathbf{x} = \mathbf{b}$ has more than one solution, then it always has infinitely many solutions.

Suppose $\mathbf{x}_1$ and $\mathbf{x}_2$ are distinct solutions of $A\mathbf{x} = \mathbf{b}$. Hence, $A\mathbf{x}_1 = A\mathbf{x}_2 = \mathbf{b}$. Then, for all real numbers t , the vector $t\mathbf{x}_1 + (1 - t)\mathbf{x}_2$ is a solution of $A\mathbf{x} = \mathbf{b}$ since

$$A(t\mathbf{x}_1 + (1 - t)\mathbf{x}_2) = t(A\mathbf{x}_1) + (1 - t)(A\mathbf{x}_2) = t\mathbf{b} + (1 - t)\mathbf{b} = \mathbf{b}.$$

Hence, $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions and the theorem is proven. ■

So our intuition on the solution sets of linear systems using lines on the $\mathbb{R}^2$ plane turns out to work for all linear systems. Now that we know how solution sets look like, we are left with the following questions:

- How do we find the solution of a linear system?

- When does a linear system have either no solution, exactly one solution, or infinitely many solutions?

The following sections will be devoted to answering these questions.

3.3 Reduced Row Echelon Form and Gauss-Jordan Elimination

In this section, we construct a formal algorithm used to find the solution of any given linear system. We begin by defining some operations used in this algorithm.

Elementary Row Operations and Elementary Matrices

Recall that linear system $A\mathbf{x} = \mathbf{b}$ can be expressed using the augmented matrix $[A | \mathbf{b}]$. On the augmented matrix, we may perform three types of **elementary row operations**:

1. $c \cdot R_i$ (Multiply c on the i th row.) ($c \neq 0$)
2. $R_i \leftrightarrow R_j$ (Exchange between i th and j th row.)
3. $cR_i + R_j$ (Add c multiplied to the i th row on to the j th row.)

Example 3.45. Suppose

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 5 \end{bmatrix}.$$

Applying row operation $3 \cdot R_1$ to A results in the matrix

$$A_1 = \begin{bmatrix} 3 & 0 & 6 \\ -1 & 3 & 5 \end{bmatrix},$$

applying $R_1 \leftrightarrow R_2$ to A results in

$$A_2 = \begin{bmatrix} -1 & 3 & 5 \\ 1 & 0 & 2 \end{bmatrix},$$

and applying $2R_1 + R_2$ to A results in

$$A_3 = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 9 \end{bmatrix}.$$

We use elementary row operations in order to turn the given matrix into a simpler form while preserving the solution set of the augmented matrix. But to show that applying elementary row operations to an augmented matrix does not affect its solution set, we must first establish the concept of elementary matrices.

Definition 3.46. An **elementary matrix** is a matrix obtained by performing a single elementary row operation to the identity matrix I_n .

Example 3.47. Elementary matrices of size 2×2 include

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix},$$

and elementary matrices of size 3×3 include

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

We use elementary matrices because they can represent elementary row operations in terms of matrix multiplication. The following theorem shows what this means.

Theorem 3.48. If elementary matrix E is an $m \times m$ matrix obtained by applying some elementary row operation to I_m and A is an $m \times n$ matrix, then EA is the matrix obtained by applying the same elementary row operation to A .

Proof. Recall that the product EA can be expressed as

$$EA = \begin{bmatrix} \mathbf{r}_1(E)A \\ \mathbf{r}_2(E)A \\ \vdots \\ \mathbf{r}_m(E)A \end{bmatrix}.$$

Using this, we shall show that the theorem works for all three types of elementary row operations.

First, consider the row operation $c \cdot R_i$. If E_1 is the elementary matrix obtained by using this row operation, then

$$\mathbf{r}_i(E_1 A) = \mathbf{r}_i(E_1)A = c\mathbf{e}_i A = c\mathbf{r}_i(A),$$

where $\mathbf{e}_i$ is the standard unit vector in $\mathbb{R}^n$ and

$$\mathbf{r}_k(E_1 A) = \mathbf{r}_k(E_1)A = \mathbf{e}_k A = \mathbf{r}_k(A) \quad (3.9)$$

for all integers $k \neq i$. Hence, $E_1 A$ is the matrix obtained by applying $c \cdot R_i$ to A .

Next, consider the row operation $R_i \leftrightarrow R_j$. If E_2 is the elementary matrix obtained with this row operation, then

$$\mathbf{r}_i(E_2 A) = \mathbf{r}_i(E_2)A = \mathbf{e}_j A = \mathbf{r}_j(A),$$

$$\mathbf{r}_j(E_2 A) = \mathbf{r}_j(E_2)A = \mathbf{e}_i A = \mathbf{r}_i(A),$$

and (3.9) works for all integers $k \neq i, j$. Hence, $E_2 A$ is the matrix obtained by applying $R_i \leftrightarrow R_j$ to A .

Finally, consider the row operation $cR_i + R_j$. If E_3 is the elementary matrix

obtained by applying this row operation, then

$$\mathbf{r}_j(E_3A) = \mathbf{r}_j(E_3)A = (c\mathbf{e}_i + \mathbf{e}_j)A = c\mathbf{r}_i(A) + \mathbf{r}_j(A),$$

and (3.9) works for all integers $k \neq j$. Hence, E_3A is the matrix obtained by applying $cR_i + R_j$ to A .

Therefore, the theorem is proven for all elementary row operations. ■

The act of applying elementary row operations to a matrix can be expressed as multiplying an elementary matrix to the left side of the given matrix. This representation holds the benefit that we can use matrix algebra to prove theorems related to elementary row operations.

We shall now prove another important property of elementary matrices that is useful for proving the theorems to come.

Theorem 3.49. *All elementary matrices are invertible, and its inverse is also an elementary matrix.*

Proof. We begin with the fact that all elementary row operations have an inverse operation. The inverse operation of $c \cdot R_i$ is $1/c \cdot R_i$, the inverse operation of $R_i \leftrightarrow R_j$ is itself, and the inverse operation of $cR_i + R_j$ is $(-c)R_i + R_j$. Applying a row operation to some matrix A followed with applying the inverse operation would result in A .

By Theorem 3.48, if E_1 is the elementary matrix obtained by a row operation and E_2 is the elementary matrix obtained by its inverse operation, then

$$E_1E_2A = E_2E_1A = A \quad \text{and} \quad E_1E_2 = E_2E_1 = I.$$

Hence, $E_2 = E_1^{-1}$ and E_1 is invertible. ■

Now we move on to proving that applying elementary row operations to an augmented matrix does not affect its solution set.

Definition 3.50. *Matrices A and B are called **row equivalent** and denoted as $A \sim B$ if either one can be obtained by applying a sequence of elementary row equations to the other matrix.*

Theorem 3.51. *If $[A \mid \mathbf{b}]$ and $[B \mid \mathbf{b}']$ are row equivalent augmented matrices, then linear systems $A\mathbf{x} = \mathbf{b}$ and $B\mathbf{x} = \mathbf{b}'$ have the same solution set.*

Proof. Claim that if E is an elementary matrix, then $A\mathbf{x} = \mathbf{b}$ and $E A\mathbf{x} = E\mathbf{b}$ have the same solution set.

Suppose $\mathbf{x}_0$ is a solution of $A\mathbf{x} = \mathbf{b}$. Then, by multiplying E to both sides of the equation $A\mathbf{x}_0 = \mathbf{b}$, we have $E A\mathbf{x}_0 = E\mathbf{b}$, which implies that $\mathbf{x}_0$ is a solution of $E A\mathbf{x} = E\mathbf{b}$.

Suppose $\mathbf{x}_0$ is a solution of $E A\mathbf{x} = E\mathbf{b}$. Then, by multiplying E^{-1} to both sides of $E A\mathbf{x}_0 = E\mathbf{b}$, we get $A\mathbf{x}_0 = \mathbf{b}$, which implies that $\mathbf{x}_0$ is a solution of $A\mathbf{x} = \mathbf{b}$. Hence, $A\mathbf{x} = \mathbf{b}$ and $E A\mathbf{x} = E\mathbf{b}$ have the same solution set.

If $[A | \mathbf{b}]$ and $[B | \mathbf{b}']$ are row equivalent, then either of the matrices can be obtained by applying a sequence of elementary row operations to the other matrix. In other words,

$$E_1 \cdots E_k [A | \mathbf{b}] = [B | \mathbf{b}'],$$

for some elementary matrices $E_1, \dots, E_k$. Since the solution set of $[A | \mathbf{b}]$ is not affected by multiplying a single elementary matrices, it would be same for a finite number of elementary matrices. Hence, $A\mathbf{x} = \mathbf{b}$ and $B\mathbf{x} = \mathbf{b}'$ have the same solution set. ■

Therefore, we can apply elementary row operations to augmented matrices without having to worry about changing the solution set.

Reduced Row Echelon Forms

Now we discuss some special matrices from which we can easily find its solution set.

Definition 3.52. *If a given matrix satisfies the following three conditions, we say that the matrix is in **row echelon form** (REF):*

- (a) *All zero rows (rows where all entries are 0) are positioned at the bottom of the matrix.*
- (b) *The leftmost nonzero entry in a nonzero row is always 1. We call this entry a **leading 1**.*
- (c) *A leading 1 must be positioned to the right of the leading 1 above it.*

*If a given matrix is in row echelon form and satisfies the following condition, we say that the matrix is in **reduced row echelon form** (RREF).*

- (d) *All entries above a leading 1 must be 0.*

Example 3.53. *The following are examples of matrices that are in row echelon form but not in reduced row echelon form:*

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The following are examples of matrices that are in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The following examples will show us how we can find the solution when we are given an augmented matrix in reduced row echelon form.

Example 3.54. Suppose we are given the augmented matrix

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

Turning the matrix into its corresponding linear system, we get

$$\begin{cases} x_1 = 3 \\ x_2 = -2 \\ x_3 = 4, \end{cases}$$

which directly leads us to the solution of the system.

Example 3.55. Augmented matrix

$$\begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

gives

$$\begin{cases} x_1 + 3x_2 + 2x_3 = -1 \\ 0 = 1, \end{cases}$$

which implies that the system has no solution since the second equation can never be true.

Example 3.56. Given an augmented matrix

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix},$$

we have

$$\begin{cases} x_1 + 2x_2 = -1 \\ x_3 = 3. \end{cases}$$

Variables that correspond to the leading 1 of the matrix are called **leading variables** and other variables are called **free variables**. In this example, x_1, x_3 would be the leading variables and x_2 would be the free variable.

When expressing the solution set, we first clear the given equations so that the left hand side of each equation is only consisted of leading variables. Hence, the equation above would look like

$$\begin{cases} x_1 = -1 - 2x_2 \\ x_3 = 3. \end{cases}$$

Next, we replace the free variables on the right with **parameters** that can be any value in $\mathbb{R}$. Hence,

$$\begin{cases} x_1 = -1 - 2t \\ x_2 = t \\ x_3 = 3, \end{cases}$$

and the solution set would look like $\{(-1 - 2t, t, 3) : t \in \mathbb{R}\}$.

The columns that correspond to leading variables are called **pivot columns**, and columns that correspond to free variables are called **nonpivot columns**.

As the above examples show, as long as we can find the reduced row echelon form of an augmented matrix, it is easy to find the solution of the given linear system. So our final step is to construct an algorithm that only uses elementary row operations to find the reduced row echelon form of the given augmented matrix.

Gauss-Jordan Elimination

The following algorithm provides a concrete, sequential method to find the reduced row echelon form of any given matrix.

Algorithm 3.57 (Gauss-Jordan Elimination). *Suppose A is a matrix. Then, we follow the following steps to find the reduced row echelon form of A :*

1. *Find the leftmost column that is not entirely consisted of zeros. If the top row of this column has a zero, then swap the top row with a row that has a nonzero entry in this column.*
2. *If the top entry in the column found in Step 1 is c , then apply $1/c \cdot R_1$ to the matrix to get a leading 1.*
3. *Apply operations $cR_1 + R_i$ for all $i \neq 1$ so that all entries below the leading 1 becomes zero.*
4. *Repeat Step 1 to 3 assuming that we have deleted the first row of the matrix. This process is to be repeated until we have the row echelon form of the matrix.*
5. *Once we have the row echelon form, start from the bottom row and apply $cR_i + R_j$ operation so that each pivot column is consisted of zeros except for the leading 1. Work all the way up to the top row to obtain the reduced row echelon form.*

Applying steps 1 to 4 in the above algorithm to matrix A results in its row echelon form. We call this process the **forward phase**. Applying up to step 5 results in the reduced row echelon form of A . We call step 5 the **backward phase** or the **back substitution** process. Steps 1 to 5 combined, we call it the **Gauss-Jordan elimination**.

Example 3.58. *Given*

$$A = \begin{bmatrix} 0 & 1 & 2 & 8 \\ 2 & -2 & 4 & 10 \\ 3 & -1 & -2 & -5 \end{bmatrix},$$

turn A into its reduced row echelon form using Algorithm 3.57.

Solution. We must go through the following steps:

1. The first column is the leftmost column with nonzero entries in this matrix. Since the top entry of this column is 0, interchange the first row with the second row so that we have a nonzero entry on the top of this column.

$$\begin{bmatrix} 0 & 1 & 2 & 8 \\ 2 & -2 & 4 & 10 \\ 3 & -1 & -2 & -5 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & 4 & 10 \\ 0 & 1 & 2 & 8 \\ 3 & -1 & -2 & -5 \end{bmatrix}$$

2. Since the top entry of the first column is 2, apply row operation $1/2 \cdot R_1$ to make a leading 1.

$$\begin{bmatrix} 2 & -2 & 4 & 10 \\ 0 & 1 & 2 & 8 \\ 3 & -1 & -2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & 1 & 2 & 8 \\ 3 & -1 & -2 & -5 \end{bmatrix}$$

3. Apply $-3R_1 + R_3$ to turn all entries below the leading 1 in the first column into 0.

$$\begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & 1 & 2 & 8 \\ 3 & -1 & -2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & -8 & -20 \end{bmatrix}$$

4. Suppose the first row has been deleted, and repeat the above process. It will result as:

$$\begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & -8 & -20 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & -12 & -36 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

5. Now that we have the row echelon form, apply back substitution to obtain the reduced row echelon form.

$$\begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Hence, via Algorithm 3.57, we have the reduced row echelon form of A :

$$R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

3.4 Linear Systems and Invertibility

Now that we know how to find the reduced row echelon form of a given matrix, we can solve any linear systems. We are finally left with one crucial question:

- When does a linear system have either no solution, exactly one solution, or infinitely many solutions?

This can be rephrased into two questions:

- When does a linear system have a solution?
- If a linear system is consistent, when is its solution unique?

Notice that we are again dealing with the existence problem and the uniqueness problem.

Existence and Uniqueness of Solution

A linear system is called **homogeneous** if $b_1 = \cdots = b_m = 0$ in (3.7). Hence, its augmented matrix would always be of the form $[A \mid \mathbf{0}]$. Homogeneous linear systems are always consistent because it has the **trivial solution** $x_1 = \cdots = x_n = 0$. Hence, a homogeneous linear system either has only the trivial solution or infinitely many solutions including the trivial solution.

Then, when does a homogeneous linear system have only the trivial solution or infinitely many solutions? Recall that in Example 3.56, the given linear system had a free variable. The free variable could be assigned to any real number, and hence the solution set would be an infinite set. This implies that a linear system has infinitely many solutions if and only if it has free variables.

Since homogeneous linear systems have either one solution or infinitely many solutions, all we have to check is whether there is a free variable or not. We can do this by looking at the reduced row echelon form of the augmented matrix. Since pivot columns represent leading variables and nonpivot columns represent free variables, the augmented matrix must not have any nonpivot columns for its corresponding homogeneous linear system to have exactly one solution. If the augmented matrix has a nonpivot column, its corresponding homogeneous linear system has infinitely many solutions.

Theorem 3.59. *If A is an $m \times n$ matrix, then the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has:*

- Exactly one solution (the trivial solution $\mathbf{x} = \mathbf{0}$) if and only if A has n pivot columns.*
- Infinitely many solutions if and only if A has less than n pivot columns.*

Example 3.60. *Consider an augmented matrix $[A \mid \mathbf{0}]$ whose reduced row echelon form is*

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Since the second column is a nonpivot column, $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions. Also, since the number of pivot columns in A can not exceed both the number

of rows and columns of A , there must be more rows than columns in A for A to have pivot columns equal to the number of its columns.

Homogeneous linear systems are significant not only because we are guaranteed the existence of its solution, but because its solution set relates to non-homogeneous linear systems too. The following theorem describes this relationship.

Theorem 3.61. *Suppose S is the solution set of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$. Then, the solution set of a consistent linear system $A\mathbf{x} = \mathbf{b}$ is*

$$\mathbf{x}_p + S = \{\mathbf{x}_p + \mathbf{x}_c : \mathbf{x}_c \in S\},$$

where $\mathbf{x}_p$ is a particular solution of $A\mathbf{x} = \mathbf{b}$.

Proof. Suppose $\mathbf{x}_c$ is an element of S . Then,

$$A(\mathbf{x}_p + \mathbf{x}_c) = A\mathbf{x}_p + A\mathbf{x}_c = \mathbf{b} + \mathbf{0} = \mathbf{b},$$

which implies that all elements in $\mathbf{x}_p + S$ is a solution of $A\mathbf{x} = \mathbf{b}$.

Now show that all solutions of $A\mathbf{x} = \mathbf{b}$ are in $\mathbf{x}_p + S$. If $\mathbf{x}_0$ is a solution of $A\mathbf{x} = \mathbf{b}$, then

$$A(\mathbf{x}_0 - \mathbf{x}_p) = A\mathbf{x}_0 - A\mathbf{x}_p = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

Hence, $\mathbf{x}_0 - \mathbf{x}_p$ is in S . Since $\mathbf{x}_0 = \mathbf{x}_p + (\mathbf{x}_0 - \mathbf{x}_p)$, we have that $\mathbf{x}_0$ is an element of $\mathbf{x}_p + S$. Therefore, $\mathbf{x}_p + S$ is a solution set of $A\mathbf{x} = \mathbf{b}$. ■

Thus, given that $A\mathbf{x} = \mathbf{b}$ is consistent, the linear system has exactly one solution if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution and the linear system has infinitely many solutions if and only if $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

All that is left to consider is when $A\mathbf{x} = \mathbf{b}$ is consistent. To solve this problem, recall Example 3.55. The reason why we concluded that the given linear system had no solution was because one of the equations derived from the reduced row echelon form turned out to be $0 = 1$. So if the $\mathbf{b}$ column in the augmented matrix $[A | \mathbf{b}]$ turns out to be a pivot column, then the corresponding system would be inconsistent, because one of the equations would turn out in the form $0 = 1$.

Theorem 3.62. *If A is an $m \times n$ matrix, then the linear system $A\mathbf{x} = \mathbf{b}$ is:*

- (a) *Consistent if and only if $\mathbf{b}$ is a pivot column in $[A | \mathbf{b}]$.*
- (b) *Inconsistent if and only if $\mathbf{b}$ is not a pivot column in $[A | \mathbf{b}]$.*

If $A\mathbf{x} = \mathbf{b}$ is consistent, then $A\mathbf{x} = \mathbf{b}$ has:

- (c) *Exactly one solution if and only if A has n pivot columns.*
- (d) *Infinitely many solutions if and only if A has less than n pivot columns.*

Using the above theorem, we can determine the existence and uniqueness of the solution of any linear systems.

Example 3.63. Consider an augmented matrix whose reduced row echelon form is

$$\begin{bmatrix} 1 & -1 & 0 & 4 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the last column is not a pivot column, the corresponding system is consistent. Also, since the second and fourth columns are nonpivot columns, the corresponding system has infinitely many solutions.

Try finding the solution set. Since

$$\begin{cases} x_1 - x_2 + 4x_4 = 2 \\ x_3 + 2x_4 = 1, \end{cases}$$

we have

$$\begin{cases} x_1 = 2 + x_2 - 4x_4 \\ x_3 = 1 - 2x_4 \end{cases} \quad \text{and} \quad \begin{cases} x_1 = 2 + s - 4t \\ x_2 = s \\ x_3 = 1 - 2t \\ x_4 = t, \end{cases}$$

which leads to the solution set $\{(2 + s - 4t, s, 1 - 2t, t) : s, t \in \mathbb{R}\}$. Hence, we have verified that the solution has infinitely many solutions.

Invertibility of Square Matrices

Now we are going to build a bridge between linear systems and matrix algebra, discussed early in the chapter. It turns out that the existence and uniqueness problems of linear systems we dealt with has linkage to the invertibility of matrices. The following theorem shows how they are linked.

Theorem 3.64. If A is an $n \times n$ matrix, then the following are equivalent:

- (a) A has a left inverse.
- (b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.
- (c) The reduced row echelon form of A is I_n .
- (d) A is a product of elementary matrices.
- (e) A is invertible.
- (f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for all $\mathbf{b} \in \mathbb{R}^n$.
- (g) A has a right inverse.

Proof. (a) $\Rightarrow$ (b) Suppose A has a left inverse B such that $BA = I_n$. Then, by multiplying B to both sides of $A\mathbf{x} = \mathbf{0}$, we get

$$BA\mathbf{x} = I_n\mathbf{x} = \mathbf{x} = \mathbf{0}.$$

Hence, $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(b) $\Rightarrow$ (c) Suppose $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Then, by Theorem 3.59, A must have n pivot columns. For A to have n pivot columns, each row and column must have a leading 1, which implies that the reduced row echelon form of A must be I_n .

(c) $\Rightarrow$ (d) If the reduced row echelon form of A is I_n , then there exists elementary matrices $E_1, E_2, \dots, E_k$ such that

$$E_1 E_2 \cdots E_k A = I_n.$$

Since $E_1, E_2, \dots, E_k$ are invertible, $E_1 E_2 \cdots E_k$ is invertible, and

$$(E_1 E_2 \cdots E_k)^{-1} = E_k^{-1} \cdots E_2^{-1} E_1^{-1}.$$

Hence,

$$A = (E_1 E_2 \cdots E_k)^{-1} I_n = E_k^{-1} \cdots E_2^{-1} E_1^{-1},$$

where $E_1^{-1}, E_2^{-1}, \dots, E_k^{-1}$ are also elementary matrices. Therefore, A is a product of elementary matrices.

(d) $\Rightarrow$ (e) Since A is a product of elementary matrices, which are invertible, A is also invertible.

(e) $\Rightarrow$ (f) If A is invertible, then $A^{-1}\mathbf{b}$ is a solution of $A\mathbf{x} = \mathbf{b}$ since

$$A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = \mathbf{b}.$$

If $\mathbf{x}_0$ is a solution of $A\mathbf{x} = \mathbf{b}$, then $A\mathbf{x}_0 = \mathbf{b}$ and

$$\mathbf{x}_0 = I_n \mathbf{x}_0 = (A^{-1}A)\mathbf{x}_0 = A^{-1}(A\mathbf{x}_0) = A^{-1}\mathbf{b}.$$

Hence, $A^{-1}\mathbf{b}$ is the unique solution of $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b} \in \mathbb{R}^n$.

(f) $\Rightarrow$ (g) If $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b} \in \mathbb{R}^m$, then there exists $\mathbf{x}_1, \dots, \mathbf{x}_n$ such that $A\mathbf{x}_1 = \mathbf{e}_1, \dots, A\mathbf{x}_n = \mathbf{e}_n$. Construct a matrix

$$C = [\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_n],$$

so that

$$AC = [A\mathbf{x}_1 \ A\mathbf{x}_2 \ \cdots \ A\mathbf{x}_n] = [\mathbf{e}_1 \ \mathbf{e}_2 \ \cdots \ \mathbf{e}_n] = I_n.$$

Hence, A has a right inverse C .

(g) $\Rightarrow$ (a) Suppose A has a right inverse C . Then, C has a left inverse A and is invertible since we have proven that (a) $\Rightarrow$ (e). Thus, there exists $n \times n$ matrix B such that $BC = CB = I_n$. Since C has a left inverse A and a right inverse B , $A = B$ by Theorem 3.26. Therefore, $CA = I_n$, meaning that A has a left inverse. ■

Theorem 3.64 is a powerful theorem that relates linear systems with matrix algebra. It also solves the problem of when a square matrix is invertible, which we were not able to solve back in Section 3.1.

In a purely algebraic perspective, the equivalence of (a), (e), and (g) implies that a square matrix A has a left inverse B if and only if it has a right inverse C , which leads to $B = C = A^{-1}$. The equivalence of (c) and (e) also tells us that by finding the reduced row echelon form of A , we can determine whether A is invertible.

In the perspective of linear systems, $A\mathbf{x} = \mathbf{b}$ is consistent and has a unique solution for all $\mathbf{b} \in \mathbb{R}^n$ when A is invertible. In fact, the unique solution can be found by the formula $\mathbf{x} = A^{-1}\mathbf{b}$, which may be a faster way to find the solution of a linear system.

Also, the proof for (c) $\Rightarrow$ (d) suggests a possible method for finding the inverse matrix of A .

Theorem 3.65. *If A is invertible, then $[A \mid I_n] \sim [I_n \mid A^{-1}]$.*

Proof. If A is invertible, there exists elementary matrices $E_1, \dots, E_k$ such that $E_1 \cdots E_k A = I_n$. Hence, $A = (E_1 \cdots E_k)^{-1}$ and $A^{-1} = E_1 \cdots E_k$. Therefore,

$$(E_1 \cdots E_k)[A \mid I_n] = [E_1 \cdots E_k A \mid E_1 \cdots E_k I_n] = [I_n \mid A^{-1}].$$

■

This method, used for finding A^{-1} , is called the **inversion algorithm**. Place matrices A and I_n next to each other like an augmented matrix, and apply row operations to the matrices so that A becomes I_n . Then, the I_n part of the matrix will become A^{-1} , due to Theorem 3.65. Obviously, this method works for all invertible square matrices.

If A is not invertible, this method will not work because A is not row equivalent to I_n according to Theorem 3.64. We can never reach I_n by applying row operations to A .

Invertibility of Nonsquare Matrices

Theorem 3.64 gave us a very deep insight into what it means for a square matrix A to be invertible. But what about nonsquare matrices? When do they hold either a left inverse or a right inverse? The following theorems give us the answer.

Theorem 3.66. *If A is an $m \times n$ matrix, then the following are equivalent:*

- (a) $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^m$.
- (b) A has m pivot columns.
- (c) A has a right inverse.

Proof. (a) $\Rightarrow$ (b) If $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^m$, then the reduced row echelon form of A must not have a zero row. Hence, each row would need to have a leading 1, implying that A has m pivot columns.

(b) $\Rightarrow$ (c) We omit this proof as it goes beyond the boundaries of this text.

(c) $\Rightarrow$ (a) Let $n \times m$ matrix C be the right inverse of A ; hence, $CA = I_n$. Then, the linear system $A\mathbf{x} = \mathbf{b}$ always has the solution $\mathbf{x} = C\mathbf{b}$ since

$$A(C\mathbf{b}) = (AC)\mathbf{b} = I_n\mathbf{b} = \mathbf{b}.$$

Therefore, $A\mathbf{x} = \mathbf{b}$ is always consistent. ■

Theorem 3.67. *If A is an $m \times n$ matrix, then the following are equivalent:*

- (a) *If $A\mathbf{x} = \mathbf{b}$ is consistent, it has exactly one solution.*
- (b) *A has n pivot columns.*
- (c) *A has a left inverse.*

Proof. (a) $\Leftrightarrow$ (b) If $A\mathbf{x} = \mathbf{b}$ has exactly one solution whenever it is consistent, then A must not have a nonpivot column. Hence, A would have n pivot columns. Conversely, if A has n pivot columns, $A\mathbf{x} = \mathbf{b}$ would have exactly one solution given that it is consistent, since it does not have any free variables.

(b) $\Rightarrow$ (c) Since A has n pivot columns, each column of the reduced row echelon form of A has a leading 1. Hence, applying elementary row operations on A , we can obtain

$$D_{m \times n} = \begin{bmatrix} I_n \\ \mathbf{0}_{(m-n) \times n} \end{bmatrix}.$$

Therefore, there exists $m \times m$ elementary matrices $E_1, \dots, E_k$ such that

$$D = E_1 \cdots E_k A.$$

Now let B be an $n \times m$ matrix such that its diagonal entries are all 1s, and other entries are all 0s. This ensures that $BD = I_n$. Therefore,

$$BE_1 \cdots E_k A = I_n,$$

which implies that if $C = BE_1 \cdots E_k$, then $CA = I_n$.

(c) $\Rightarrow$ (a) Let $n \times m$ matrix B be the left inverse of A . Suppose that the linear system $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}_0$. Then, $A\mathbf{x}_0 = \mathbf{b}$ and

$$\mathbf{x}_0 = I_n \mathbf{x}_0 = (BA)\mathbf{x}_0 = B(A\mathbf{x}_0) = B\mathbf{b}.$$

Therefore, if $A\mathbf{x} = \mathbf{b}$ has a solution, its solution is unique. ■

Theorem 3.66 states the equivalence conditions for when the existence of solution is guaranteed. That A has m pivot columns implies $m \leq n$, which leads to either a square matrix or a nonsquare matrix where $m < n$. If $m < n$, we say that the linear system is **underdetermined**, since there are more variables than equations.

Theorem 3.67 states the equivalence conditions for when the uniqueness of solution is guaranteed. That A has n pivot columns implies $m \geq n$, which leads to either a square matrix or a nonsquare matrix where $m > n$. If $m > n$, we say

that the linear system is **overdetermined**, since there are more equations than variables.

Since a nonsquare matrix that has a left inverse must satisfy $m \geq n$ and one that has a right inverse must satisfy $m \leq n$, a nonsquare matrix can not have both a left inverse and a right inverse. Also, notice that if A is an $n \times n$ square matrix with n pivot columns, then all six statements in the two theorems would be equivalent to each other. Hence, it would lead us back to the results of Theorem 3.64.

3.5 Fundamental Subspaces of Matrices

In this section, we observe four fundamental subspaces that originate from a given matrix A : the row space, column space, null space of A , and null space of A^T . We shall discuss how these spaces arise, what they imply, how to find their bases, and their relations.

Row and Column Spaces

Suppose we are given a linear system $A\mathbf{x} = \mathbf{b}$, where A is an $m \times n$ matrix. The term $A\mathbf{x}$ can be expressed as

$$\begin{aligned} A\mathbf{x} &= [\mathbf{c}_1(A) \quad \mathbf{c}_2(A) \quad \cdots \quad \mathbf{c}_n(A)] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &= x_1\mathbf{c}_1(A) + x_2\mathbf{c}_2(A) + \cdots + x_n\mathbf{c}_n(A), \end{aligned}$$

which is the linear combination of the column vectors of A . So, that $A\mathbf{x} = \mathbf{b}$ is consistent means that there exists $x_1, \dots, x_n \in \mathbb{R}$ such that

$$x_1\mathbf{c}_1(A) + x_2\mathbf{c}_2(A) + \cdots + x_n\mathbf{c}_n(A) = \mathbf{b}.$$

Using what we studied in Section 2.3, we could rephrase it as: $A\mathbf{x} = \mathbf{b}$ is consistent if $\mathbf{b}$ is a vector in $\text{span}\{\mathbf{c}_1(A), \mathbf{c}_2(A), \dots, \mathbf{c}_n(A)\}$.

Hence the vector space $\text{span}\{\mathbf{c}_1(A), \mathbf{c}_2(A), \dots, \mathbf{c}_n(A)\}$ is a subspace of $\mathbb{R}^m$ that includes all $\mathbf{b}$ that makes $A\mathbf{x} = \mathbf{b}$ consistent. We shall give this subspace a special name.

Definition 3.68. If A is an $m \times n$ matrix, then the **column space** of A , denoted as $\text{col}(A)$, is the span of its column vectors. Hence,

$$\text{col}(A) = \text{span}\{\mathbf{c}_1(A), \mathbf{c}_2(A), \dots, \mathbf{c}_n(A)\}.$$

The column space can also be expressed as

$$\text{col}(A) = \{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\},$$

since $A\mathbf{x}$ is a linear combination of the column vectors of A . The following theorem summarizes the properties of column spaces that we have investigated above.

Theorem 3.69. *Let A be an $m \times n$ matrix. Then:*

- (a) $\text{col}(A)$ is a subspace of $\mathbb{R}^m$.
- (b) $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b} \in \text{col}(A)$.

In a similar context, we may also define a subspace spanned by the row vectors of A . Hence we give the following definition.

Definition 3.70. *If A is an $m \times n$ matrix, then the **row space** of A , denoted as $\text{row}(A)$, is the span of its row vectors. Hence,*

$$\text{row}(A) = \text{span}\{\mathbf{r}_1(A), \mathbf{r}_2(A), \dots, \mathbf{r}_m(A)\}.$$

Notice that $\text{row}(A) = \text{col}(A^T)$, since the row vectors of A are column vectors of A^T . Hence, the row space has to do with the consistency of the linear system $A^T\mathbf{x} = \mathbf{b}$. The following theorem summarizes this.

Theorem 3.71. *Let A be an $m \times n$ matrix. Then:*

- (a) $\text{row}(A) = \text{col}(A^T)$.
- (a) $\text{row}(A)$ is a subspace of $\mathbb{R}^n$.
- (b) $A^T\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b} \in \text{row}(A)$.

Null Spaces

Consider a homogeneous linear system $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix. We found that its solution set is either empty, $\{\mathbf{0}\}$, or an infinite set of solutions. But it turns out that the solution set is actually a vector space. We prove this fact below.

Theorem 3.72. *The solution set of $A\mathbf{x} = \mathbf{0}$ is a vector space.*

Proof. Suppose A is an $m \times n$ matrix. Then, $S = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$ is a subset of $\mathbb{R}^n$. Use Theorem 2.13 to show that S is a subspace of $\mathbb{R}^n$.

If $\mathbf{x}_1, \mathbf{x}_2 \in S$, then $A\mathbf{x}_1 = A\mathbf{x}_2 = \mathbf{0}$. Hence,

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0},$$

which implies that $\mathbf{x}_1 + \mathbf{x}_2 \in S$.

If $\mathbf{x}_1 \in S$ and $c \in \mathbb{R}$, then $A\mathbf{x}_1 = \mathbf{0}$ and

$$A(c\mathbf{x}_1) = c(A\mathbf{x}_1) = c\mathbf{0} = \mathbf{0},$$

which implies that $c\mathbf{x}_1 \in S$.

Finally, $\mathbf{0}$ is a trivial solution, so it is always in S . Therefore, S is a subspace of $\mathbb{R}^n$. ■

Note that the solution set of linear systems like $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} \neq \mathbf{0}$ is not a vector space. As we saw in Theorem 3.61, the solution set of $A\mathbf{x} = \mathbf{b}$ looks like $\mathbf{x}_p + S$, where $\mathbf{x}_p$ is a particular solution of $A\mathbf{x} = \mathbf{b}$ and S is the solution set of $A\mathbf{x} = \mathbf{0}$. Hence, the zero vector $\mathbf{0}$ can not be in the set $\mathbf{x}_p + S$, and $\mathbf{x}_p + S$ can not be a vector space. We discussed in Example 2.16 that such spaces are called Affine spaces.

Anyways, the solution set of a homogeneous linear system is a vector space, so we may as well give a special name for this set too.

Definition 3.73. If A is an $m \times n$ matrix, then the **null space** of A , denoted as $\text{null}(A)$, is the solution set of $A\mathbf{x} = \mathbf{0}$. Hence,

$$\text{null}(A) = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}.$$

The row space, column space, null space of A , and null space of A^T combined are called the four **fundamental subspaces** of A . If A is an $m \times n$ matrix, its row space and null space of A are subspaces of $\mathbb{R}^n$ and its column space and null space of A^T are subspaces of $\mathbb{R}^m$. In the following pages, we discuss how these subspaces relate to each other.

Finding the Bases

Before we relate the fundamental subspaces to each other, we must first discuss how to find the basis of these subspaces explicitly, in order to discuss their bases and dimensions.

We start with the easiest one of all: the null space. We already studied how to find the null space A when some $m \times n$ matrix A is given. So we know the procedure: use Gauss-Jordan elimination, find the reduced row echelon form, and find all the solutions. All that we have to do is find the basis from the general solution. The below example illustrates how this is done.

Example 3.74. Suppose A is a 4×5 matrix given as

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 1 & -2 & 2 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 2 & -4 & 1 & 1 & 1 \end{bmatrix}.$$

Using Gauss-Jordan elimination, we can find the reduced row echelon form of A , which turns out to be

$$R = \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Turning $R\mathbf{x} = \mathbf{0}$ into the form of equations, we have

$$\begin{cases} x_1 = 2x_2 + x_4 \\ x_3 = -3x_4 \\ x_5 = 0 \end{cases}$$

and hence,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 \\ x_2 \\ -3x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} = x_2 \mathbf{v}_1 + x_4 \mathbf{v}_2,$$

where

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore,

$$\text{null}(A) = \{c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 : c_1, c_2 \in \mathbb{R}\},$$

which implies that $\text{null}(A) = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is also linearly independent, a basis for $\text{null}(A)$ would be $\{\mathbf{v}_1, \mathbf{v}_2\}$.

So we can find the basis for null space by expressing the general solution in its parameter form and extracting the vectors involved in the parameter form. Basis for null space of A^T can also be found in a similar way, except that we would have to consider the matrix A^T .

Now we move onto finding the basis for the row space. We start with the following theorem, which gives us a hint of how this is done.

Theorem 3.75. *If $A \sim B$, then $\text{row}(A) = \text{row}(B)$.*

Proof. Suppose $A = CB$, where C is a product of elementary matrices. Then, each row of B would be an element of $\text{row}(A)$, since they can be expressed as a linear combination of the rows of A .

Let $\mathbf{x}$ be a vector in $\text{row}(B)$. Since each row of B is in $\text{row}(A)$, it is obvious that $\mathbf{x}$ is in $\text{row}(A)$.

Similarly, we can show that if $\mathbf{x}$ is in $\text{row}(A)$, then $\mathbf{x}$ is in $\text{row}(B)$ because $B = C^{-1}A$ where C^{-1} is also a product of elementary matrices. Hence, $\text{row}(A) = \text{row}(B)$. ■

So we can apply row operations to a given matrix A without worrying it affecting the row space. If R is the reduced row echelon form of A , then $\text{row}(R) = \text{row}(A)$ and the basis for $\text{row}(R)$ becomes the basis for $\text{row}(A)$. Since the nonzero rows of R becomes the basis for $\text{row}(R)$ (why?), the nonzero rows of R also becomes

the basis for $\text{row}(A)$. Hence, we can find the basis for the row space. The below example illustrates this process.

Example 3.76. *Reconsider the matrix A in Example 3.74. The reduced row echelon form R was*

$$R = \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and hence, a basis for $\text{row}(A)$ would be $\{\mathbf{r}_1(R), \mathbf{r}_2(R), \mathbf{r}_3(R)\}$, because it spans $\text{row}(A)$ (because $\text{row}(A) = \text{row}(R)$) and is linearly independent (due to the leading 1s).

Finally, how do we find the basis for the column space? First of all, we can not use the method used for finding the basis for the row space, because applying row operations to the given matrix may change the column space, in general (why?). But instead, the following theorem gives us a different method for finding the basis for the column space.

Theorem 3.77. *If A and B are $m \times n$ matrices such that $A \sim B$, then a set of column vectors of A forms the basis for $\text{col}(A)$ if and only if the set of corresponding column vectors of B forms the basis for $\text{col}(B)$.*

Proof. Let

$$A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n] \quad \text{and} \quad B = [\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_n],$$

and construct new matrices

$$A' = [\mathbf{a}_{i_1} \ \mathbf{a}_{i_2} \ \cdots \ \mathbf{a}_{i_k}] \quad \text{and} \quad B' = [\mathbf{b}_{i_1} \ \mathbf{b}_{i_2} \ \cdots \ \mathbf{b}_{i_k}],$$

where $i_1, \dots, i_k$ are integers between 1 and n . Also, let $A = CB$ and $A' = CB'$, where C is a product of elementary matrices.

First, claim that $\text{col}(A) = \text{col}(A')$ if and only if $\text{col}(B) = \text{col}(B')$. In other words, claim that

$$\{A'\mathbf{x} : \mathbf{x} \in \mathbb{R}^k\} = \{A\mathbf{y} : \mathbf{y} \in \mathbb{R}^n\}$$

if and only if

$$\{B'\mathbf{x} : \mathbf{x} \in \mathbb{R}^k\} = \{B\mathbf{y} : \mathbf{y} \in \mathbb{R}^n\}.$$

If $\text{col}(A) = \text{col}(A')$, then it is obvious that $\text{col}(B')$ is a subset of $\text{col}(B)$, since columns of B include the columns of B' . Also, if $B\mathbf{y}$ is a vector in $\text{col}(B)$, then for such $\mathbf{y} \in \mathbb{R}^n$, there exists $\mathbf{x} \in \mathbb{R}^k$ such that $A'\mathbf{x} = A\mathbf{y}$, since $\text{col}(A) = \text{col}(A')$. Hence, $C^{-1}A'\mathbf{x} = C^{-1}A\mathbf{y}$ and $B'\mathbf{x} = B\mathbf{y}$. Therefore, $\text{col}(B)$ is a subset of $\text{col}(B')$ and hence, $\text{col}(B) = \text{col}(B')$.

Conversely, suppose $\text{col}(B) = \text{col}(B')$. Then $\text{col}(A')$ is obviously a subset of $\text{col}(A)$. Also, if $A\mathbf{y}$ is a vector in $\text{col}(A)$, then there exists $\mathbf{x} \in \mathbb{R}^k$ such that $B'\mathbf{x} = B\mathbf{y}$, since $\text{col}(B) = \text{col}(B')$. Hence, $CB'\mathbf{x} = CB\mathbf{y}$ and $A'\mathbf{x} = A\mathbf{y}$. Therefore, $\text{col}(A) = \text{col}(A')$.

Next, claim that columns of A' are linearly independent if and only if columns of B' are linearly independent. Suppose $A'\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$. The linear system $B'\mathbf{x} = \mathbf{0}$ can be turned into $CB'\mathbf{x} = C\mathbf{0} = \mathbf{0}$ and $A'\mathbf{x} = \mathbf{0}$, which implies that $B'\mathbf{x} = \mathbf{0}$ also has only the trivial solution.

Conversely, if $B'\mathbf{x} = \mathbf{0}$ has only the trivial solution, then $A'\mathbf{x} = \mathbf{0}$ also has only the trivial solution since it can be turned into $C^{-1}A'\mathbf{x} = C^{-1}\mathbf{0} = \mathbf{0}$ and $B'\mathbf{x} = \mathbf{0}$. Therefore, $A'\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if $B'\mathbf{x} = \mathbf{0}$ has only the trivial solution, which implies that columns of A' are linearly independent if and only if columns of B' are linearly independent.

Since columns of A' are linearly independent and span $\text{col}(A)$ if and only if columns of B' are linearly independent and span $\text{col}(B)$, the theorem is proven. ■

Suppose A has a reduced row echelon form R . Since the pivot columns of R form the basis for $\text{col}(R)$, by the above theorem., the corresponding pivot columns of A form the basis for $\text{col}(A)$. We look at an example of this process below.

Example 3.78. Recall the matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 1 & -2 & 2 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 2 & -4 & 1 & 1 & 1 \end{bmatrix}$$

in Example 3.74 again. Since the reduced row echelon form of A was

$$R = \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

the pivot columns of A are the first, third, and fifth columns. Thus, the first, third, and fifth columns of A form a basis for $\text{col}(A)$.

Rank and Nullity

Now that we know how to find the basis for all fundamental subspaces of a matrix, we may discuss their dimensions. The dimensions of these subspaces are given a special name as follows.

Definition 3.79. Let A be an $m \times n$ matrix. The **rank** of A is the dimension of its column space and the **nullity** of A is the dimension of the null space of A . Hence,

$$\text{rank}(A) = \dim(\text{col}(A)) \quad \text{and} \quad \text{nullity}(A) = \dim(\text{null}(A)).$$

Following the above definition, the dimension of the row space and the null space

of A^T can be expressed as

$$\text{rank}(A^T) = \dim(\text{row}(A)) \quad \text{and} \quad \text{nullity}(A^T) = \dim(\text{null}(A^T)).$$

Let us consider what the rank is in various contexts. We saw that the basis for the column space of A is formed by the pivot columns of A . Hence, the rank of A is the number of pivot columns in A . It would also mean the number of leading variables in $A\mathbf{x} = \mathbf{0}$, and the number of leading 1s in its reduced row echelon form R .

In a similar context, the nullity of A is the number of nonpivot columns in A , and the number of free variables in $A\mathbf{x} = \mathbf{0}$.

Recall that the basis for the row space of A was formed by the nonzero rows of R . Hence, the dimension of $\text{row}(A)$ would be the number of leading 1s in R , which is equal to $\text{rank}(A)$ itself. Therefore, the dimension of the row space and the column space are actually the same.

Theorem 3.80. *If A is an $m \times n$ matrix, then $\text{rank}(A) = \text{rank}(A^T)$.*

The next theorem is one of the fundamental results of linear algebra that shows the relation between rank and nullity.

Theorem 3.81. *If A is an $m \times n$ matrix, then*

$$\text{rank}(A) + \text{nullity}(A) = n.$$

Proof. The rank of A is the number of pivot columns in A , and the nullity of A is the number of nonpivot columns in A . Hence, the sum of rank and nullity is the number of all columns in A , which is n . ■

Example 3.82. *In Example 3.74, 3.76, and 3.78, we found the basis for $\text{null}(A)$, $\text{row}(A)$, and $\text{col}(A)$. Hence, the dimension of $\text{null}(A)$ was 2, and the dimensions of $\text{row}(A)$ and $\text{col}(A)$ were 3 and 3 (since the row space and column space has the same dimension). So we can check that*

$$\text{rank}(A) + \text{nullity}(A) = 3 + 2 = 5,$$

as Theorem 3.81 suggests.

Now under this interpretation, we may extend the equivalent statements in Theorem 3.66 and Theorem 3.67.

Theorem 3.83. *If A is an $m \times n$ matrix, then the following are equivalent:*

- (a) $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^m$.
- (b) A has m pivot columns.
- (d) $\text{rank}(A) = m$.
- (e) $\text{col}(A) = \mathbb{R}^m$.

Proof. (b) $\Leftrightarrow$ (d) Since $\text{rank}(A)$ is the number of pivot columns in A , (b) and (d) are equivalent.

(d) $\Leftrightarrow$ (e) By Theorem 3.69 (a), $\text{col}(A)$ is a subspace of $\mathbb{R}^m$. Hence, by Theorem 2.49 (c), $\text{col}(A) = \mathbb{R}^m$ if and only if $\text{rank}(A) = m$. ■

The above theorem shows the conditions for when a linear system is guaranteed the existence of solution. Recall that $\text{col}(A)$ was the set of all vectors in $\mathbb{R}^m$ that made $A\mathbf{x} = \mathbf{b}$ consistent. Hence it would require $\text{col}(A)$ to be $\mathbb{R}^m$ in order for $A\mathbf{x} = \mathbf{b}$ to be consistent for all $\mathbf{b}$ in $\mathbb{R}^m$.

Theorem 3.84. *If A is an $m \times n$ matrix, then the following are equivalent:*

- (a) *If $A\mathbf{x} = \mathbf{b}$ is consistent, it has exactly one solution.*
- (b) *A has n pivot columns.*
- (d) $\text{rank}(A) = n$.
- (e) $\text{row}(A) = \mathbb{R}^n$.
- (f) $\text{null}(A) = \{\mathbf{0}\}$.
- (g) *Column vectors of A are linearly independent.*

Proof. (b) $\Leftrightarrow$ (d) Since $\text{rank}(A)$ is the number of pivot columns in A , (b) and (d) are equivalent.

(d) $\Leftrightarrow$ (e) By Theorem 3.71 (b), $\text{row}(A)$ is a subspace of $\mathbb{R}^n$. Hence, by Theorem 2.49 (c), $\text{row}(A) = \mathbb{R}^n$ if and only if $\text{rank}(A) = n$.

(d) $\Leftrightarrow$ (f) $\text{rank}(A) = n$ if and only if $\text{nullity}(A) = 0$ due to Theorem 3.81. And $\text{nullity}(A) = 0$ if and only if $\text{null}(A) = \{\mathbf{0}\}$ by definition.

(f) $\Leftrightarrow$ (g) Column vectors of A are linearly independent if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, since $A\mathbf{x}$ is the linear combination of column vectors of A . ■

Above theorem shows the conditions for when a linear system is guaranteed the uniqueness of solution. Intuitively, $\text{null}(A)$ must be a zero vector space for $A\mathbf{x} = \mathbf{b}$ to have at most one solution since the solution set of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x}_p + \text{null}(A)$ by Theorem 3.61.

Combining the two theorems above, we may extend Theorem 3.64 too.

Theorem 3.85. *If A is an $n \times n$ matrix, then the following are equivalent:*

- (e) *A is invertible.*
- (h) *Row vectors (or column vectors) of A are linearly independent.*
- (i) *Row vectors (or column vectors) of A spans $\mathbb{R}^n$.*
- (j) *Row vectors (or column vectors) of A form a basis for $\mathbb{R}^n$.*
- (k) $\text{row}(A) = \text{col}(A) = \mathbb{R}^n$.

- (l) $\text{rank}(A) = n$.
- (m) $\text{null}(A) = \{\mathbf{0}\}$.
- (n) $\text{nullity}(A) = 0$.

This theorem, considering all the statements from (a) to (n), builds a bridge between matrix algebra, linear systems, and vector spaces. In the coming chapters, we will continue building this theorem, to create more bridges with other core topics of linear algebra.

4 | Linear Maps

We shall now focus on some special functions called linear maps. Linear maps are the generalized version of the linear function $y = ax$, which is represented by a line. Although some linear maps may not be visualized, they still hold the same algebraic properties as other functions with linearity.

Under this general notion of functions, we examine the theories of linear maps in detail, starting from kernels and images to the rank and nullity of linear maps. We will also see how matrices can represent linear maps, and how matrix algebra relates with linear maps. The concept of isomorphism and invertibility is especially important. Finally, we visit the concept of similarity, where a fundamental result of linear maps awaits.

4.1 Linear Maps Between Vector Spaces

The word "linear" has to do with straight lines in a geometric sense. For example, consider a function $f : \mathbb{R} \rightarrow \mathbb{R}$ that maps x to ax . Its graph would be the line $y = ax$ above the xy -plane. Then, the function f would preserve addition and scalar multiplication in the sense that

$$f(x_1 + x_2) = f(x_1) + f(x_2) \quad \text{and} \quad f(cx_1) = cf(x_1)$$

for all x_1, x_2 , and $c \in \mathbb{R}$. The general notion of linear maps has to do with these two properties too. Hence, we give the following definition.

Definition 4.1. Suppose $T : V \rightarrow W$ is a function where V and W are vector spaces. If:

- (a) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$,
- (b) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u} \in V$ and $c \in \mathbb{R}$,

then T is called a **linear map**, or we say that T is linear.

Example 4.2. Let V, W be vector spaces and define a function $T : V \rightarrow W$ such that $T(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v}$ in V . Then, T is linear since

$$T(\mathbf{u} + \mathbf{v}) = \mathbf{0} = \mathbf{0} + \mathbf{0} = T(\mathbf{u}) + T(\mathbf{v})$$

and

$$T(c\mathbf{u}) = \mathbf{0} = c\mathbf{0} = cT(\mathbf{u})$$

for all $\mathbf{u}, \mathbf{v}$ in V and c in $\mathbb{R}$. We call T the **zero map**.

Example 4.3. Let V be a vector space, and define a function $I_V : V \rightarrow V$ such that $I_V(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v}$ in V . Then, it can be shown that I_V is linear, via Definition 4.1. We call I_V the **identity map** over V .

Example 4.4. Let $C^\infty(\mathbb{R})$ be the set of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ that is infinitely differentiable on $\mathbb{R}$ (which is a vector space). If $D : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$ is a function that maps f to f' , then D is linear since

$$D(f + g) = (f + g)' = f' + g' = D(f) + D(g)$$

and

$$D(cf) = (cf)' = cf' = cD(f)$$

for all $f, g \in C^\infty(\mathbb{R})$ and scalar $c \in \mathbb{R}$. Note that D is the differential operator.

Example 4.5. Suppose $T : M_{m \times n} \rightarrow M_{n \times m}$ such that $T(A) = A^T$. Then, it can be shown that T is linear using Definition 4.1.

Example 4.6. Recall that $C(\mathbb{R})$ denotes the set of all real-valued functions continuous on $\mathbb{R}$. A function $T : C(\mathbb{R}) \rightarrow C(\mathbb{R})$ defined as

$$T(f) = \int_0^x f(t) dt$$

for all $f \in C(\mathbb{R})$ is linear since

$$T(f + g) = \int_0^x (f(t) + g(t)) dt = \int_0^x f(t) dt + \int_0^x g(t) dt = T(f) + T(g)$$

and

$$T(cf) = \int_0^x cf(t) dt = c \int_0^x f(t) dt = cT(f)$$

for all $f, g \in C(\mathbb{R})$ and $c \in \mathbb{R}$.

From Definition 4.1, we can derive the following common properties of linear maps.

Theorem 4.7. Let V, W be vector spaces. If $T : V \rightarrow W$ is linear, then:

- (a) $T(\mathbf{0}) = \mathbf{0}$.
- (b) $T(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + \cdots + c_nT(\mathbf{v}_n)$.

Proof. (a) Since $0\mathbf{v} = \mathbf{0}$ for all $\mathbf{v}$ in V due to Theorem 2.10 (a),

$$T(\mathbf{0}) = T(0\mathbf{v}) = 0 \cdot T(\mathbf{v}) = \mathbf{0}.$$

(b) Using Definition 4.1, we have

$$\begin{aligned} T(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) &= T(c_1\mathbf{v}_1) + \cdots + T(c_n\mathbf{v}_n) \\ &= c_1T(\mathbf{v}_1) + \cdots + c_nT(\mathbf{v}_n). \end{aligned}$$

■

Another useful property of linear maps is that linear maps can be uniquely determined by the image of a basis. In other words, linear maps are dependent on the image of a basis. The below theorem explains this in more detail.

Theorem 4.8. *Let V, W be vector spaces and $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for V . Suppose $T : V \rightarrow W$ is a function defined as*

$$T(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) = c_1\mathbf{w}_1 + \cdots + c_n\mathbf{w}_n \quad (4.1)$$

for all scalars $c_1, \dots, c_n$ in $\mathbb{R}$, where $\mathbf{w}_1, \dots, \mathbf{w}_n$ are some vectors in W . Then T is the only linear map such that $T(\mathbf{v}_i) = \mathbf{w}_i$ for all integers i between 1 and n .

Proof. First check that T is linear. This is left as **Exercise**. Next, check that T defined as (4.1) satisfies $T(\mathbf{v}_i) = \mathbf{w}_i$ for all i . This can be done by substituting appropriate values for $c_1, \dots, c_n$.

Finally, suppose $S : V \rightarrow W$ is a linear map such that $S(\mathbf{v}_i) = \mathbf{w}_i$ for all i . Then, for some $\mathbf{v}$ in V such that $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$,

$$S(\mathbf{v}) = c_1S(\mathbf{v}_1) + \cdots + c_nS(\mathbf{v}_n) = c_1\mathbf{w}_1 + \cdots + c_n\mathbf{w}_n = T(\mathbf{v}),$$

which implies that $S = T$. Hence, T is unique. ■

Kernel and Image

Now we look at two types of sets that are defined for a given linear map: the kernel and the image. These sets arise purely from the context of linear maps, and help us analyze the structure of a linear map more precisely.

Definition 4.9. *Let $T : V \rightarrow W$ be linear, where V, W are vector spaces. The **kernel** of T , denoted as $\ker(T)$, is defined as*

$$\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\},$$

*and the **image** of T , denoted as $\text{im}(T)$, is defined as*

$$\text{im}(T) = \{T(\mathbf{v}) \in W : \mathbf{v} \in V\}.$$

Theorem 4.10. *Suppose $T : V \rightarrow W$ is linear, where V, W are vector spaces. Then:*

- (a) $\ker(T)$ is a subspace of V .

(b) $\text{im}(T)$ is a subspace of W .

Proof. (a) It is obvious from the definition that $\ker(T)$ is a subset of V . Hence, use the subspace test to show that $\ker(T)$ is a subspace of V . If $\mathbf{v}_1, \mathbf{v}_2$ are vectors in $\ker(T)$ and c is a scalar in $\mathbb{R}$, then

$$T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2) = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

and

$$T(c\mathbf{v}_1) = cT(\mathbf{v}_1) = c\mathbf{0} = \mathbf{0},$$

which implies that $\mathbf{v}_1 + \mathbf{v}_2$ and $c\mathbf{v}_1$ are also elements in $\ker(T)$. Also, $\mathbf{0}$ is in $\ker(T)$ since $T(\mathbf{0}) = \mathbf{0}$. Therefore, $\ker(T)$ is a subspace of V .

(b) We can see from the definition that $\text{im}(T)$ is a subset of W . Now suppose $\mathbf{w}_1, \mathbf{w}_2$ are vectors in $\text{im}(T)$ and c is a scalar in $\mathbb{R}$. Then there exists $\mathbf{v}_1, \mathbf{v}_2$ in V such that

$$T(\mathbf{v}_1) = \mathbf{w}_1 \quad \text{and} \quad T(\mathbf{v}_2) = \mathbf{w}_2.$$

Hence,

$$T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2) = \mathbf{w}_1 + \mathbf{w}_2$$

and

$$T(c\mathbf{v}_1) = cT(\mathbf{v}_1) = c\mathbf{w}_1,$$

which implies that $\mathbf{w}_1 + \mathbf{w}_2$ and $c\mathbf{w}_1$ are also elements in $\text{im}(T)$. Also, $\mathbf{0}$ is in $\text{im}(T)$ since $T(\mathbf{0}) = \mathbf{0}$. Therefore, $\text{im}(T)$ is a subspace of W . ■

Since $\ker(T)$ and $\text{im}(T)$ are vector spaces, it is natural that we take interest in their bases and dimension. When we discuss the dimension of these spaces, we use a special term as defined in the following definition.

Definition 4.11. Let $T : V \rightarrow W$ be linear, where V, W are vector spaces. The **rank** of T , denoted as $\text{rank}(T)$, is defined as

$$\text{rank}(T) = \dim(\ker(T)),$$

and the **nullity** of T , denoted as $\text{nullity}(T)$, is defined as

$$\text{nullity}(T) = \dim(\text{im}(T)).$$

Recall that the rank and nullity of a matrix was the dimension of the column space and the null space of a given matrix. The rank and nullity of a linear map, however, refers to the dimension of the kernel and the image. But fundamentally, they hold the same meaning. We will see why this is in the next section.

For now, we shall try proving a fundamental result for linear maps analogous to Theorem 3.81. The following theorem serves as a useful tool to do this.

Theorem 4.12. Let $T : V \rightarrow W$ be linear, where V, W are vector spaces. If

$\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for V , then

$$\text{im}(T) = \text{span}\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}.$$

Proof. Theorem 2.28 implies that $\text{span}\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ is the smallest subspace of W that contains $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$. Hence, $\text{im}(T)$ contains the span since $\mathbf{v}_i \in \text{im}(T)$ for all integers i .

Now suppose $\mathbf{w} \in \text{im}(T)$. Then there exists $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$ and $\mathbf{v} = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$ for some $c_1, \dots, c_n \in \mathbb{R}$. Hence,

$$\mathbf{w} = T(\mathbf{v}) = T(c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + \dots + c_nT(\mathbf{v}_n),$$

which implies that $\mathbf{w}$ is in $\text{span}\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$. Thus, the theorem is proven. ■

Now we head to the fundamental theorem of linear maps. It gives the relation between the rank and nullity of linear map T .

Theorem 4.13. *Let $T : V \rightarrow W$ be linear, where V is a finite-dimensional vector space and W is a vector space too. Then,*

$$\text{rank}(T) + \text{nullity}(T) = \dim(V).$$

Proof. Let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ be a basis for $\ker(T)$ and $\mathcal{B}' = \mathcal{B} \cup \{\mathbf{v}'_1, \dots, \mathbf{v}'_n\}$ be a basis for V extended from $\mathcal{B}$, since $\ker(T)$ is a subspace of V .

Claim that $S = \{T(\mathbf{v}'_1), \dots, T(\mathbf{v}'_n)\}$ is a basis for $\text{im}(T)$. First, show that S spans $\text{im}(T)$. $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_m), T(\mathbf{v}'_1), \dots, T(\mathbf{v}'_n)\}$ spans $\text{im}(T)$ due to Theorem 4.12 and $T(\mathbf{v}_1) = \dots = T(\mathbf{v}_m) = \mathbf{0}$ because $\mathbf{v}_1, \dots, \mathbf{v}_m \in \ker(T)$. Hence, S spans $\text{im}(T)$.

Next, show that S is linearly independent. Suppose $c_1T(\mathbf{v}'_1) + \dots + c_nT(\mathbf{v}'_n) = \mathbf{0}$ where $c_1, \dots, c_n$ are scalars in $\mathbb{R}$. Then,

$$T(c_1\mathbf{v}'_1 + \dots + c_n\mathbf{v}'_n) = \mathbf{0}$$

and $c_1\mathbf{v}'_1 + \dots + c_n\mathbf{v}'_n$ is in $\ker(T)$. Since $\mathcal{B}$ is a basis for $\ker(T)$, there exists $d_1, \dots, d_m \in \mathbb{R}$ such that $c_1\mathbf{v}'_1 + \dots + c_n\mathbf{v}'_n = d_1\mathbf{v}_1 + \dots + d_m\mathbf{v}_m$. And because $\mathcal{B}'$ is linearly independent, we have $c_1 = \dots = c_n = d_1 = \dots = d_m = 0$, which implies that S is linearly independent.

Therefore, S is a basis for $\text{im}(T)$ and

$$\text{rank}(T) + \text{nullity}(T) = m + n = \dim(V).$$
■

A very naive way of interpreting Theorem 4.13 is by thinking that $\ker(T)$ is the set of all vectors that makes the image $\mathbf{0}$ and $\text{im}(T)$ is the set of all vectors that can be an image. For any vector $\mathbf{w}$ in W , the set of all vectors that makes the image $\mathbf{w}$ has equal size as $\ker(T)$ since T is linear. Hence, the size of $\ker(T)$ and $\text{im}(T)$

multiplied should equal to the size of V , since V is the domain of T . In terms of dimensions, multiplying the size equals adding the dimensions, so it makes sense that $\text{rank}(T) + \text{nullity}(T) = \dim(V)$.

Operations with Linear Maps

Definition 4.14. Suppose V, W are vector spaces. Then, $\mathcal{L}(V, W)$ denotes the set of all linear transformations from V to W . If $V = W$, we may write $\mathcal{L}(V, W)$ as $\mathcal{L}(V)$, and elements of $\mathcal{L}(V)$ are called **linear operators** over V .

We may define addition and scalar multiplication on $\mathcal{L}(V, W)$ in a pointwise manner, such that if $S, T \in \mathcal{L}(V, W)$ and $c \in \mathbb{R}$, then

$$(S + T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v})$$

and

$$(cT)(\mathbf{v}) = cT(\mathbf{v})$$

for all $\mathbf{v}$ in V . Based on this definition, we prove the following theorem.

Theorem 4.15. If V, W are vector spaces, then $(\mathcal{L}(V, W), +, \cdot)$ is a vector space, where operations $+$ and $\cdot$ are defined in a pointwise manner.

Proof. Claim that $\mathcal{L}(V, W)$ is a vector space using the vector space axioms. First, show that $\mathcal{L}(V, W)$ is closed under addition. If $S, T \in \mathcal{L}(V, W)$, then

$$\begin{aligned} (S + T)(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) &= S(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) + T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) \\ &= c_1S(\mathbf{v}_1) + c_2S(\mathbf{v}_2) + c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) \\ &= c_1(S + T)(\mathbf{v}_1) + c_2(S + T)(\mathbf{v}_2), \end{aligned}$$

which implies that $S + T$ is linear, and is in $\mathcal{L}(V, W)$. Also, claim that $\mathcal{L}(V, W)$ is closed under scalar multiplication. If $T \in \mathcal{L}(V, W)$ and $c \in \mathbb{R}$, then

$$\begin{aligned} (cT)(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) &= cT(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) \\ &= c(c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2)) \\ &= c_1(cT)(\mathbf{v}_1) + c_2(cT)(\mathbf{v}_2), \end{aligned}$$

which implies that cT is linear and is in $\mathcal{L}(V, W)$. Showing the rest of the vector space axioms is left as **Exercise**. ■

We do not go further into analyzing how the vector space $\mathcal{L}(V, W)$ looks like, because this is beyond the boundaries of our interest. For now, we shall continue our discussion on the basic properties of linear maps.

4.2 Matrix Representation of Linear Maps

In the previous chapter, we introduced the concept of matrices, which was used to express linear systems and vectors in a simple manner. This section is devoted

to showing how matrices are used to express linear maps. In fact, matrices are a powerful tool for understanding the characteristics of linear maps.

We recall on the notion of coordinate vectors. If V is a vector space and $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis for V , then the coordinate vector of vector $\mathbf{v}$ in V relative to basis $\mathcal{B}$ is

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

given that $\mathbf{v} = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$. We said that coordinate vectors are useful because it allows any abstract vector in a finite-dimensional vector space to be expressed as a vector in the familiar $\mathbb{R}^n$ space.

Also, coordinate vectors preserve linearity. In other words, if $\Phi_{\mathcal{B}} : V \rightarrow \mathbb{R}^n$ is a function such that $\Phi_{\mathcal{B}}(\mathbf{v}) = [\mathbf{v}]_{\mathcal{B}}$, then $\Phi_{\mathcal{B}}$ is a linear map, since

$$\Phi_{\mathcal{B}}(\mathbf{v}_1 + \mathbf{v}_2) = [\mathbf{v}_1 + \mathbf{v}_2]_{\mathcal{B}} = [\mathbf{v}_1]_{\mathcal{B}} + [\mathbf{v}_2]_{\mathcal{B}} = \Phi_{\mathcal{B}}(\mathbf{v}_1) + \Phi_{\mathcal{B}}(\mathbf{v}_2)$$

and

$$\Phi_{\mathcal{B}}(c\mathbf{v}_1) = [c\mathbf{v}_1]_{\mathcal{B}} = c[\mathbf{v}_1]_{\mathcal{B}} = c\Phi_{\mathcal{B}}(\mathbf{v}_1)$$

for all $\mathbf{v}_1, \mathbf{v}_2$ in V and scalar c in $\mathbb{R}$. $\Phi_{\mathcal{B}}$ is called the **coordinate map** of V with respect to basis $\mathcal{B}$. Since the coordinate vector uniquely exists for all vectors in V , $\Phi_{\mathcal{B}}$ is bijective too.

Now back to the original question: how can we use matrices to represent a linear map? A matrix can only express vectors in $\mathbb{R}^n$, so to deal with linear maps that map between abstract vectors, we have to change the vectors into their coordinate vectors.

$$\begin{array}{ccc} (V, \mathcal{B}) & \xrightarrow{T} & (W, \mathcal{B}') \\ \downarrow \Phi_{\mathcal{B}} & & \downarrow \Phi_{\mathcal{B}'} \\ \mathbb{R}^n & \xrightarrow{\mathcal{B}'[T]_{\mathcal{B}}} & \mathbb{R}^m \end{array} \quad \begin{array}{ccc} \mathbf{v} & \xrightarrow{T} & T(\mathbf{v}) \\ \downarrow \Phi_{\mathcal{B}} & & \downarrow \Phi_{\mathcal{B}'} \\ [\mathbf{v}]_{\mathcal{B}} & \xrightarrow{\mathcal{B}'[T]_{\mathcal{B}}} & [T(\mathbf{v})]_{\mathcal{B}'} \end{array}$$

The diagrams above illustrates our idea. Suppose V, W are finite-dimensional vector spaces with bases $\mathcal{B}, \mathcal{B}'$, respectively. Let $\dim(V) = n$ and $\dim(W) = m$. Also, let $\Phi_{\mathcal{B}} : V \rightarrow \mathbb{R}^n$ be a linear map such that $\Phi_{\mathcal{B}}(\mathbf{v}) = [\mathbf{v}]_{\mathcal{B}}$ and similarly, $\Phi_{\mathcal{B}'} : W \rightarrow \mathbb{R}^m$ be a linear map such that $\Phi_{\mathcal{B}'}(\mathbf{w}) = [\mathbf{w}]_{\mathcal{B}'}$.

If $T : V \rightarrow W$ is a linear map that maps $\mathbf{v}$ to $T(\mathbf{v})$, then with the help of $\Phi_{\mathcal{B}}$ and $\Phi_{\mathcal{B}'}$, we can obtain $[\mathbf{v}]_{\mathcal{B}}$ and $[T(\mathbf{v})]_{\mathcal{B}'}$. Now that we have coordinate vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$, all we have to do is find some matrix A that relates $[\mathbf{v}]_{\mathcal{B}}$ and $[T(\mathbf{v})]_{\mathcal{B}'}$ such that

$$A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{B}'}.$$

Suppose $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $\mathcal{B}' = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$. Then,

$$[\mathbf{v}_1]_{\mathcal{B}} = \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad [\mathbf{v}_n]_{\mathcal{B}} = \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

and

$$A[\mathbf{v}_1]_{\mathcal{B}} = \mathbf{c}_1(A), \quad \dots, \quad A[\mathbf{v}_n]_{\mathcal{B}} = \mathbf{c}_n(A).$$

Hence, we have

$$A = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{B}'} & \dots & [T(\mathbf{v}_n)]_{\mathcal{B}'} \end{bmatrix},$$

an $m \times n$ matrix. We denote A as ${}_{\mathcal{B}'}[T]_{\mathcal{B}}$ and call it the matrix for T with respect to the bases $\mathcal{B}$ and $\mathcal{B}'$, so we get $[T(\mathbf{v})]_{\mathcal{B}'} = {}_{\mathcal{B}'}[T]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}}$.

In some sense, it is obvious that matrix ${}_{\mathcal{B}'}[T]_{\mathcal{B}}$ is capable of expressing T since we saw in Theorem 4.8 that a linear map is uniquely defined when we are given the image of all vectors in a basis for V .

Note that if $V = W$ and $\mathcal{B} = \mathcal{B}'$, then we denote ${}_{\mathcal{B}'}[T]_{\mathcal{B}}$ as $[T]_{\mathcal{B}}$ which is an $n \times n$ matrix that looks like

$$[T]_{\mathcal{B}} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{B}} & \dots & [T(\mathbf{v}_n)]_{\mathcal{B}} \end{bmatrix}$$

and satisfies

$$[T]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{B}}.$$

Example 4.16. Suppose $T : \mathbb{P}_2 \rightarrow \mathbb{P}_2$ is a linear map such that

$$T(c_0 + c_1t + c_2t^2) = 2c_0 + c_1t + 3c_2t^2.$$

If $\mathcal{B} = \{1, t, t^2\}$, then

$$[T(1)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad [T(t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad [T(t^2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

which gives

$$[T]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

We can check that if $p(t) = c_0 + c_1t + c_2t^2$ for some scalars $c_0, c_1, c_2 \in \mathbb{R}$, then

$$[T]_{\mathcal{B}}[p]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2c_0 \\ c_1 \\ 3c_2 \end{bmatrix} = [T(p)]_{\mathcal{B}}.$$

With the matrix representation in hand, we revisit the kernel and range of a linear

map. Since the matrix ${}_{\mathcal{B}'}[T]_{\mathcal{B}}$ satisfies

$${}_{\mathcal{B}'}[T]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{B}'},$$

we can redefine the kernel and range of T as

$$\ker(T) = \{\mathbf{v} \in V : {}_{\mathcal{B}'}[T]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = \mathbf{0}\}$$

and

$$\operatorname{im}(T) = \{\mathbf{w} \in W : [\mathbf{w}]_{\mathcal{B}'} = {}_{\mathcal{B}'}[T]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}}, \forall \mathbf{v} \in V\}.$$

Notice that the null space and column space of ${}_{\mathcal{B}'}[T]_{\mathcal{B}}$,

$$\operatorname{null}({}_{\mathcal{B}'}[T]_{\mathcal{B}}) = \{\mathbf{x} \in \mathbb{R}^n : {}_{\mathcal{B}'}[T]_{\mathcal{B}}\mathbf{x} = \mathbf{0}\}$$

and

$$\operatorname{col}({}_{\mathcal{B}'}[T]_{\mathcal{B}}) = \{{}_{\mathcal{B}'}[T]_{\mathcal{B}}\mathbf{x} \in \mathbb{R}^m : \mathbf{x} \in \mathbb{R}^n\},$$

are pretty much the same as the kernel and the range of T . So the two concepts end up having the same meaning. This also explains why Theorem 3.81 and Theorem 4.13 looked alike.

Now we move onto observing how matrix algebra relates to linear maps. This will be our focus throughout the rest of this chapter.

Theorem 4.17. *Let V, W be vector spaces with bases $\mathcal{B}, \mathcal{B}'$, respectively, and S, T be linear maps in $\mathcal{L}(V, W)$. Then:*

- (a) ${}_{\mathcal{B}'}[S + T]_{\mathcal{B}} = {}_{\mathcal{B}'}[S]_{\mathcal{B}} + {}_{\mathcal{B}'}[T]_{\mathcal{B}}$.
- (b) ${}_{\mathcal{B}'}[cT]_{\mathcal{B}} = c {}_{\mathcal{B}'}[T]_{\mathcal{B}}$ for all $c \in \mathbb{R}$.

Proof. **Exercise.** ■

Composite Maps

We now observe the composite of linear maps, which also turns out to be a linear map. Also, we shall show that composite maps can be expressed as matrix products. First, a brief definition on composite maps.

Definition 4.18. *Suppose $T_1 : U \rightarrow V$ and $T_2 : V \rightarrow W$ are linear maps, where U, V, W are vector spaces. Then, the **composite map** $T_2 \circ T_1 : U \rightarrow W$ is a function such that*

$$(T_2 \circ T_1)(\mathbf{u}) = T_2(T_1(\mathbf{u}))$$

for all $\mathbf{u}$ in U .

Composite maps can be defined for a finite number of linear maps in a similar sense. For example, the composite map $T_3 \circ T_2 \circ T_1$ would be a function defined as

$$(T_3 \circ T_2 \circ T_1)(\mathbf{u}) = T_3(T_2(T_1(\mathbf{u}))),$$

where T_1, T_2, T_3 are the appropriate linear maps.

Next, we show that the composite of linear maps is also linear.

Theorem 4.19. *If $T_1 : U \rightarrow V$ and $T_2 : V \rightarrow W$ are linear maps where U, V, W are vector spaces, then $T_2 \circ T_1$ is also a linear map.*

Proof. Since

$$\begin{aligned} (T_2 \circ T_1)(\mathbf{u} + \mathbf{v}) &= T_2(T_1(\mathbf{u} + \mathbf{v})) = T_2(T_1(\mathbf{u}) + T_1(\mathbf{v})) \\ &= T_2(T_1(\mathbf{u})) + T_2(T_1(\mathbf{v})) = (T_2 \circ T_1)(\mathbf{u}) + (T_2 \circ T_1)(\mathbf{v}) \end{aligned}$$

and

$$\begin{aligned} (T_2 \circ T_1)(c\mathbf{u}) &= T_2(T_1(c\mathbf{u})) = T_2(cT_1(\mathbf{u})) \\ &= cT_2(T_1(\mathbf{u})) = c(T_2 \circ T_1)(\mathbf{u}) \end{aligned}$$

for all $\mathbf{u}, \mathbf{v}$ in U and scalar c in $\mathbb{R}$, $T_2 \circ T_1$ is linear. ■

Since the composite map $T_2 \circ T_1$ is also a linear map, we may find the matrix for $T_2 \circ T_1$. Suppose $T_1 : U \rightarrow V$ and $T_2 : V \rightarrow W$ are linear maps and U, V, W are vector spaces with bases $\mathcal{B}, \mathcal{B}', \mathcal{B}''$, respectively. If $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$, $\mathcal{B}' = \{\mathbf{v}_1, \dots, \mathbf{v}_r\}$, and $\mathcal{B}'' = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$, then

$${}_{\mathcal{B}'}[T_1]_{\mathcal{B}} = \begin{bmatrix} [T(\mathbf{u}_1)]_{\mathcal{B}'} & \cdots & [T(\mathbf{u}_n)]_{\mathcal{B}'} \end{bmatrix}$$

and

$${}_{\mathcal{B}''}[T_2]_{\mathcal{B}'} = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{B}''} & \cdots & [T(\mathbf{v}_r)]_{\mathcal{B}''} \end{bmatrix}.$$

Hence,

$${}_{\mathcal{B}'}[T_1]_{\mathcal{B}}[\mathbf{u}]_{\mathcal{B}} = [T_1(\mathbf{u})]_{\mathcal{B}'} \quad \text{and} \quad {}_{\mathcal{B}''}[T_2]_{\mathcal{B}'}[\mathbf{v}]_{\mathcal{B}'} = [T_2(\mathbf{v})]_{\mathcal{B}''},$$

which leads to

$$\begin{aligned} {}_{\mathcal{B}''}[T_2]_{\mathcal{B}'}[T_1]_{\mathcal{B}}[\mathbf{u}]_{\mathcal{B}} &= {}_{\mathcal{B}''}[T_2]_{\mathcal{B}'}[T_1(\mathbf{u})]_{\mathcal{B}'} \\ &= [T_2(T_1(\mathbf{u}))]_{\mathcal{B}''} = [(T_2 \circ T_1)(\mathbf{u})]_{\mathcal{B}''} = {}_{\mathcal{B}''}[T_2 \circ T_1]_{\mathcal{B}}[\mathbf{u}]_{\mathcal{B}}. \end{aligned}$$

Therefore, we have the following theorem.

Theorem 4.20. *If $T_1 : U \rightarrow V$ and $T_2 : V \rightarrow W$ are linear maps where U, V, W are vector spaces with bases $\mathcal{B}, \mathcal{B}', \mathcal{B}''$, respectively, then*

$${}_{\mathcal{B}''}[T_2 \circ T_1]_{\mathcal{B}} = {}_{\mathcal{B}''}[T_2]_{\mathcal{B}'}[T_1]_{\mathcal{B}}.$$

The matrix for a composite map can thus be obtained via matrix multiplication. This also implies that the algebraic properties of matrices work similarly for linear maps. For example, if T_1, T_2, T_3 are linear maps defined so that the below operations can be defined, then:

- $T_1 \circ (T_2 + T_3) = (T_1 \circ T_2) + (T_1 \circ T_3).$

- $(T_1 \circ T_2) \circ T_3 = T_1 \circ (T_2 \circ T_3)$.
- $(aT_1) \circ T_2 = T_1 \circ (aT_2)$.

Theorem 4.20 also lets us think about why we defined the matrix multiplication in a rather strange way back in Chapter 3.

For AB to be defined, it required the number of columns of A and the number of rows of B to be same. This is the same as that for $T_2 \circ T_1$ to be defined, the codomain of T_1 and the domain of T_2 have to be the same.

4.3 Isomorphism and Invertibility

When we deal with functions in general, we ask the question of whether the given function is one-to-one, onto, or bijective. Hence, we should do this with linear maps too. We start by reviewing what it means for a function to be one-to-one or onto.

Definition 4.21. A linear map $T : V \rightarrow W$, where V, W are vector spaces, is **one-to-one** if $T(\mathbf{v}_1) = T(\mathbf{v}_2)$ implies $\mathbf{v}_1 = \mathbf{v}_2$ for all vectors $\mathbf{v}_1, \mathbf{v}_2$ in V . T is **onto** if there exists $\mathbf{v}$ in V such that $T(\mathbf{v}) = \mathbf{w}$ for all $\mathbf{w}$ in W .

The following theorems give the equivalent statements for a linear map to be either one-to-one or onto.

Theorem 4.22. If $T : V \rightarrow W$ is linear and V, W are vector spaces, then the following are equivalent:

- (a) T is one-to-one.
- (b) $\ker(T) = \{\mathbf{0}\}$.

Proof. (a) $\Rightarrow$ (b) Since T is one-to-one, $\mathbf{0}$ is the only vector such that $T(\mathbf{0}) = \mathbf{0}$. Hence, $\ker(T) = \{\mathbf{0}\}$.

(b) $\Rightarrow$ (a) Let $\mathbf{v}_1, \mathbf{v}_2$ be vectors in V such that $T(\mathbf{v}_1) = T(\mathbf{v}_2)$. Then,

$$T(\mathbf{v}_1) - T(\mathbf{v}_2) = T(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}$$

and hence, $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}$ because $\ker(T) = \{\mathbf{0}\}$. Since $\mathbf{v}_1 = \mathbf{v}_2$, we have that T is one-to-one. ■

Theorem 4.23. If $T : V \rightarrow W$ is linear and V, W are vector spaces, then the following are equivalent:

- (a) T is onto.
- (b) $\text{im}(T) = W$.

Proof. The definition of onto implies that (a) and (b) are equivalent. ■

If $\dim(V) = \dim(W)$ in the above two theorems, we get to combine the above two theorems, which yields the following result.

Theorem 4.24. *If $T : V \rightarrow W$ is linear and V, W are vector spaces such that $\dim(V) = \dim(W)$, then the following are equivalent:*

- (a) T is one-to-one.
- (b) $\ker(T) = \{\mathbf{0}\}$ (i.e., $\text{nullity}(T) = 0$).
- (c) $\text{im}(T) = W$ (i.e., $\text{rank}(T) = \dim(W) = \dim(V)$).
- (d) T is onto.

Proof. (b) $\Rightarrow$ (c) If $\ker(T) = \{\mathbf{0}\}$, then $\text{nullity}(T) = 0$. By Theorem 4.13, we have

$$\text{rank}(T) = \dim(V) = \dim(W),$$

which implies that $\text{im}(T) = W$, due to Theorem 2.49 (c).

(c) $\Rightarrow$ (b) If $\text{im}(T) = W$, then $\text{rank}(T) = \dim(W) = \dim(V)$. Again, using Theorem 4.13, we have

$$\text{nullity}(T) = \dim(V) - \text{rank}(T) = 0,$$

which implies that $\ker(T) = \{\mathbf{0}\}$. ■

In the theory of functions, a function that is both one-to-one and onto is called bijective. But in the theory of linear maps, we give a different name for such functions.

Definition 4.25. *If $T : V \rightarrow W$ is one-to-one and onto, then T is called an **isomorphism**. Also, if there exists an isomorphism between vector spaces V and W , we say that V and W are **isomorphic**.*

The term isomorphic is used to indicate that the vector spaces V and W have the same algebraic structure. But what does it mean for two vector spaces to have the same algebraic structure?

Theorem 4.26. *Suppose V and W are finite-dimensional vector spaces. Then, V and W are isomorphic if and only if $\dim(V) = \dim(W)$.*

Proof. ($\Rightarrow$) If V, W are isomorphic, then there exists isomorphism $T : V \rightarrow W$. Since T is one-to-one and onto, $\text{rank}(T) = \dim(W)$ and $\text{nullity}(T) = 0$ by Theorem 4.22 and 4.23. Hence, $\text{rank}(T) + \text{nullity}(T) = \dim(W) = \dim(V)$, due to Theorem 4.13.

($\Leftarrow$) Suppose $\dim(V) = \dim(W) = n$ and $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}, \mathcal{B}' = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ are bases for V and W , respectively. Define a unique linear map $T : V \rightarrow W$ such that $T(\mathbf{v}_i) = \mathbf{w}_i$ for all integers i , in reference to Theorem 4.8. Then,

$$\text{im}(T) = \text{span}\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\} = \text{span}\{\mathbf{w}_1, \dots, \mathbf{w}_n\} = W,$$

which implies that T is onto. Hence, by Theorem 4.24, T is an isomorphism and V, W are isomorphic. ■

So the algebraic structure of a vector space is determined by its dimension. The elements that make up a vector space is not important. It is the dimension of the vector space that decides how the vector space is structured. This shows how important dimensions are in the context of vector spaces.

Corollary 4.27. *Vector space V is isomorphic to $\mathbb{R}^n$ if and only if $\dim(V) = n$.*

The above is a corollary of Theorem 4.26. An example of an isomorphism between n -dimensional vector space V and $\mathbb{R}^n$ is the coordinate map $\Phi_B : V \rightarrow \mathbb{R}^n$ which maps vector $\mathbf{v}$ in V to the coordinate vector $[\mathbf{v}]_B$ in $\mathbb{R}^n$. Φ_B is also called the **natural isomorphism** from V to $\mathbb{R}^n$.

Inverse Maps and Invertibility

Now, we move onto the inverse map of linear maps. Inverse maps are analogous to inverse functions, so we give the following definition.

Definition 4.28. *Let $T : V \rightarrow W$ be linear, where V, W are vector spaces. If there exists $T^{-1} : W \rightarrow V$ is a function such that $T^{-1} \circ T = I_V$ and $T \circ T^{-1} = I_W$, where I_V, I_W are identity maps, then we say that T is invertible and T^{-1} is called the **inverse** of T .*

Recall that inverse functions can be defined if and only if the given function is bijective. Hence, we are led to the following theorem.

Theorem 4.29. *Linear map T is invertible if and only if T is an isomorphism.*

So our discussions on isomorphism are connected to inverse maps. The following theorem lists some basic properties of inverse maps.

Theorem 4.30. *If $S : U \rightarrow V$ and $T : V \rightarrow W$ are invertible linear maps, then:*

- (a) T^{-1} is linear.
- (a) $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$.
- (b) $(T^{-1})^{-1} = T$.

Proof. (a) If $\mathbf{v}_1, \mathbf{v}_2$ are vectors in V and $\mathbf{w}_1, \mathbf{w}_2$ are vectors in W such that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$, then

$$\begin{aligned} T^{-1}(\mathbf{w}_1 + \mathbf{w}_2) &= T^{-1}(T(\mathbf{v}_1) + T(\mathbf{v}_2)) \\ &= T^{-1}(T(\mathbf{v}_1 + \mathbf{v}_2)) = \mathbf{v}_1 + \mathbf{v}_2 = T^{-1}(\mathbf{w}_1) + T^{-1}(\mathbf{w}_2) \end{aligned}$$

and

$$\begin{aligned} T^{-1}(c\mathbf{w}_1) &= T^{-1}(cT(\mathbf{v}_1)) \\ &= T^{-1}(T(c\mathbf{v}_1)) = c\mathbf{v}_1 = cT^{-1}(\mathbf{w}_1) \end{aligned}$$

for all $c \in \mathbb{R}$. Therefore, T^{-1} is linear. ■

Proof to (b) and (c) are left as **Exercise**.

The following theorem relates the invertibility of matrices with the invertibility of linear maps, forming a bridge between matrix algebra and linear maps. It also provides a formula for the matrix of an inverse map.

Theorem 4.31. *Suppose $T : V \rightarrow W$ is a linear map, where V, W are vector spaces with bases $\mathcal{B}, \mathcal{B}'$, respectively. Then, the following are equivalent:*

- (a) T is invertible.
- (b) ${}_{\mathcal{B}'}[T]_{\mathcal{B}}$ is invertible.

Proof. ($\Rightarrow$) Suppose T is invertible. Then T is an isomorphism between V and W , implying that $\dim(V) = \dim(W) = n$. Also, there exists $T^{-1} : W \rightarrow V$ such that $T^{-1} \circ T = I_V$ and $T \circ T^{-1} = I_W$. Hence,

$${}_{\mathcal{B}'}[T^{-1}]_{\mathcal{B}'}[T]_{\mathcal{B}} = [T^{-1} \circ T]_{\mathcal{B}} = [I_V]_{\mathcal{B}} = I_n$$

and

$${}_{\mathcal{B}'}[T]_{\mathcal{B}}[T^{-1}]_{\mathcal{B}'} = [T \circ T^{-1}]_{\mathcal{B}'} = [I_W]_{\mathcal{B}'} = I_n,$$

implying that ${}_{\mathcal{B}'}[T]_{\mathcal{B}}$ is invertible and

$$({}_{\mathcal{B}'}[T]_{\mathcal{B}})^{-1} = {}_{\mathcal{B}}[T^{-1}]_{\mathcal{B}'}.$$

($\Leftarrow$) Suppose ${}_{\mathcal{B}'}[T]_{\mathcal{B}}$ is invertible. Then, suppose the inverse matrix is a matrix for some linear map S , and show that $S \circ T = I_V$, $T \circ S = I_W$. Rest of the proof is left as **Exercise**. ■

So the invertibility of linear maps and its matrix is an equivalent statement, and the inverse matrix of T is the matrix of the inverse map T^{-1} . This is obvious, since the matrix of the identity map is the inverse matrix, and matrix multiplication implies composite maps.

The above theorem allows us to extend Theorem 3.85, so that it includes the context of linear maps.

Theorem 4.32. *If A is an $n \times n$ matrix and $T : V \rightarrow W$ is a linear map such that $A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{B}'}$ for all $\mathbf{v}$ in V , where V, W are vector spaces with bases $\mathcal{B}, \mathcal{B}'$, respectively, then the following are equivalent:*

- (e) A is invertible.
- (o) $\ker(T) = \{\mathbf{0}\}$.
- (p) $\text{im}(T) = W$.
- (q) T is invertible.

From what we have learned so far, we can sense that matrix algebra and linear maps are intricately related to each other. Matrices suggest an elegant way of analyzing linear maps, and make it easier for us to find composite maps and inverse maps.

4.4 Similarity

Consider $T : V \rightarrow W$, where V, W are vector spaces. Recall that if $\mathcal{B}, \mathcal{C}$ are some bases for V and W respectively, then

$${}_c[T]_{\mathcal{B}} = [T(\mathbf{v}_1)]_{\mathcal{C}} \quad \cdots \quad [T(\mathbf{v}_n)]_{\mathcal{C}}$$

is a matrix of T with respect to bases $\mathcal{B}$ and $\mathcal{C}$. So the matrix of T can actually change according to which bases for V and W are chosen.

But one might ask: is there any relationship between matrices of the same linear map, even though its basis may be different? In other words, if $\mathcal{B}'$ and $\mathcal{C}'$ are another bases for V and W respectively, then what is the relationship between ${}_c[T]_{\mathcal{B}}$ and ${}_{c'}[T]_{\mathcal{B}'}$?

Transition Matrices

Before we deal with our question, we have to go over a special type of map. Consider the diagram below.

$$\begin{array}{ccc} (V, \mathcal{B}) & \xrightarrow{I_V} & (V, \mathcal{B}') \\ \downarrow \Phi_{\mathcal{B}} & & \downarrow \Phi_{\mathcal{B}'} \\ \mathbb{R}^n & \xrightarrow{{}_{\mathcal{B}'}[I_V]_{\mathcal{B}}} & \mathbb{R}^n \end{array} \quad \begin{array}{ccc} \mathbf{v} & \xrightarrow{I_V} & \mathbf{v} \\ \downarrow \Phi_{\mathcal{B}} & & \downarrow \Phi_{\mathcal{B}'} \\ [\mathbf{v}]_{\mathcal{B}} & \xrightarrow{{}_{\mathcal{B}'}[I_V]_{\mathcal{B}}} & [\mathbf{v}]_{\mathcal{B}'} \end{array}$$

V is a vector space with bases $\mathcal{B}, \mathcal{B}'$, and the map $I_V : V \rightarrow V$ is the identity map, which maps $\mathbf{v}$ into itself. But since the basis changes, $[\mathbf{v}]_{\mathcal{B}}$ and $[\mathbf{v}]_{\mathcal{B}'}$ are not the same. Hence, we have to find the matrix ${}_{\mathcal{B}'}[I_V]_{\mathcal{B}}$, which gives ${}_{\mathcal{B}'}[I_V]_{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{B}'}$.

Let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, $\mathcal{B}' = \{\mathbf{v}'_1, \dots, \mathbf{v}'_n\}$, and suppose $\mathbf{v} = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$ for some scalars $c_1, \dots, c_n$ in $\mathbb{R}$. Then

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \quad \text{and} \quad [\mathbf{v}]_{\mathcal{B}'} = c_1[\mathbf{v}_1]_{\mathcal{B}'} + \dots + c_n[\mathbf{v}_n]_{\mathcal{B}'},$$

which leads to

$$\begin{bmatrix} [\mathbf{v}_1]_{\mathcal{B}'} & \cdots & [\mathbf{v}_n]_{\mathcal{B}'} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = [\mathbf{v}]_{\mathcal{B}'}.$$

Hence,

$${}_{\mathcal{B}'}[I_V]_{\mathcal{B}} = \begin{bmatrix} [\mathbf{v}_1]_{\mathcal{B}'} & \cdots & [\mathbf{v}_n]_{\mathcal{B}'} \end{bmatrix}.$$

We also denote ${}_{\mathcal{B}'}[I_V]_{\mathcal{B}}$ as $P_{\mathcal{B} \rightarrow \mathcal{B}'}$, and call it the **transition matrix** from $\mathcal{B}$ to $\mathcal{B}'$. An important characteristic of transition matrices is that

$$(P_{\mathcal{B} \rightarrow \mathcal{B}'})(P_{\mathcal{B}' \rightarrow \mathcal{B}})[\mathbf{v}]_{\mathcal{B}'} = (P_{\mathcal{B} \rightarrow \mathcal{B}'})[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{B}'}$$

and

$$(P_{\mathcal{B}' \rightarrow \mathcal{B}})(P_{\mathcal{B} \rightarrow \mathcal{B}'})[\mathbf{v}]_{\mathcal{B}} = (P_{\mathcal{B}' \rightarrow \mathcal{B}})[\mathbf{v}]_{\mathcal{B}'} = [\mathbf{v}]_{\mathcal{B}},$$

which implies that

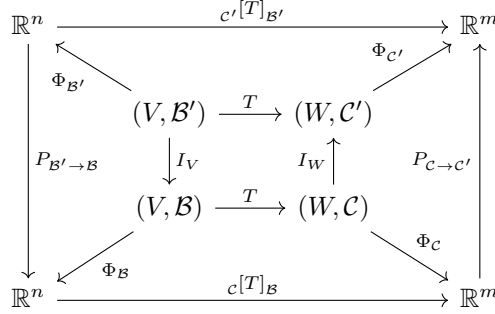
$$(P_{\mathcal{B} \rightarrow \mathcal{B}'})(P_{\mathcal{B}' \rightarrow \mathcal{B}}) = (P_{\mathcal{B}' \rightarrow \mathcal{B}})(P_{\mathcal{B} \rightarrow \mathcal{B}'}) = I$$

and $P_{\mathcal{B}' \rightarrow \mathcal{B}}, P_{\mathcal{B} \rightarrow \mathcal{B}'}$ are invertible. Hence, we have proven the following theorem.

Theorem 4.33. *All transition matrices are invertible.*

Similar Matrices

Now back to our question. What is the relationship between ${}_c[T]_{\mathcal{B}}$ and ${}_{c'}[T]_{\mathcal{B}'}$? Suppose $\dim(V) = n$ and $\dim(W) = m$, and consider the following diagram.



The diagram suggests that

$${}_{c'}[T]_{\mathcal{B}'}[\mathbf{v}]_{\mathcal{B}'} = (P_{\mathcal{C} \rightarrow \mathcal{C}'}){}_c[T]_{\mathcal{B}}(P_{\mathcal{B}' \rightarrow \mathcal{B}})[\mathbf{v}]_{\mathcal{B}'} = [T(\mathbf{v})]_{\mathcal{C}'}$$

and

$${}_{c'}[T]_{\mathcal{B}'} = (P_{\mathcal{C} \rightarrow \mathcal{C}'}){}_c[T]_{\mathcal{B}}(P_{\mathcal{B}' \rightarrow \mathcal{B}}) \quad (4.2)$$

as a result. Therefore, we can see that ${}_c[T]_{\mathcal{B}}$ and ${}_{c'}[T]_{\mathcal{B}'}$ are relatable using the transition matrices.

Consider the special case where $V = W$, $\mathcal{B} = \mathcal{C}$, and $\mathcal{B}' = \mathcal{C}'$. Then, (4.2) becomes

$$\begin{aligned} [T]_{\mathcal{B}'} &= (P_{\mathcal{B} \rightarrow \mathcal{B}'}) [T]_{\mathcal{B}} (P_{\mathcal{B}' \rightarrow \mathcal{B}}) \\ &= (P_{\mathcal{B}' \rightarrow \mathcal{B}})^{-1} [T]_{\mathcal{B}} (P_{\mathcal{B}' \rightarrow \mathcal{B}}), \end{aligned}$$

and substitute $P = P_{\mathcal{B}' \rightarrow \mathcal{B}}$ to get

$$[T]_{\mathcal{B}'} = P^{-1} [T]_{\mathcal{B}} P.$$

Definition 4.34. *Suppose A and B are square matrices. If there exists invertible matrix P such that $B = P^{-1}AP$, then we say that A and B are **similar**.*

Hence, $[T]_{\mathcal{B}'}$ and $[T]_{\mathcal{B}}$ are similar. Similar matrices share various properties in common, such as its rank and nullity. We will prove this in Chapter 6. For now, take it for granted that although $[T]_{\mathcal{B}'}$ and $[T]_{\mathcal{B}}$ are different matrices, they share similar properties as a matrix. This implies that no matter what basis we choose, the matrices of T represent a fundamentally equivalent linear map.

5 | Determinants

In this chapter, we look at a special function that maps $M_{n \times n}(\mathbb{R})$ to $\mathbb{R}$, called the determinant. So the determinant assigns some real number to all square matrices, whose value holds significance for both linear systems and linear maps.

We will see what the determinant exactly means, how it is defined, how it is calculated, its properties, and its linkage to linear algebra.

5.1 Definition of Determinants

Consider a linear map $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$. If an n -dimensional parallelepiped determined by the vectors $T(\mathbf{e}_1), \dots, T(\mathbf{e}_n)$ has an n -dimensional volume of c , then for any region D in $\mathbb{R}^n$, its image $T(D)$ would have an n -dimensional volume of $c \cdot m(D)$, where $m(D)$ is the n -dimensional volume of D .

The value c is the ratio between $m(T(D))$ and $m(D)$, and is the volume of the parallelepiped determined by $T(\mathbf{e}_1), \dots, T(\mathbf{e}_n)$, since the parallelepiped is the image of an n -dimensional cube determined by the unit vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$, which has an n -dimensional volume of 1.

If we can find this c value, it would turn out to be very convenient, since we can find the volume of any image of some given region D in the domain of T .

This is where the determinant comes in. We want the determinant to be a function that assigns each square matrix of linear operator T to the c value. Since

$$[T]_{\mathcal{E}} = [[T(\mathbf{e}_1)]_{\mathcal{E}} \ [T(\mathbf{e}_2)]_{\mathcal{E}} \ \cdots \ [T(\mathbf{e}_n)]_{\mathcal{E}}],$$

where $\mathcal{E}$ is the standard basis for $\mathbb{R}^n$, the determinant of $[T]_{\mathcal{E}}$ would be the same as the volume of the parallelepiped formed by the column vectors of $[T]_{\mathcal{E}}$. Hence, we give the following definition.

Definition 5.1. The *determinant* is a function $\det : M_{n \times n}(\mathbb{R}) \rightarrow \mathbb{R}$ such that:

- (a) $\det[\cdots c\mathbf{v}_i + \mathbf{w}_i \cdots] = c \cdot \det[\cdots \mathbf{v}_i \cdots] + \det[\cdots \mathbf{w}_i \cdots]$.
- (b) $\det[\cdots \mathbf{v}_i \cdots \mathbf{v}_j \cdots] = -\det[\cdots \mathbf{v}_j \cdots \mathbf{v}_i \cdots]$.
- (c) $\det(I_n) = 1$.

Remark. The determinant of A can be denoted as either $\det(A)$ or $|A|$.

First of all, notice that $\det$ is defined only for square matrices. This is because nonsquare matrices deal with linear maps between vector spaces of different dimensions. It would be unnecessary to deal with such linear maps because we can not discuss the volume ratio between regions from different dimensions.

(a) is a basic property that $\det$ has to satisfy in order to represent the volume of a parallelepiped. If one of the column vectors is scaled, the volume would be scaled by the same number, and if one of the column vectors is added by another vector, the volume would be the sum of the volumes when either one of the two vectors forms a parallelepiped.

(b) is used to assign the orientation of the basis. It does not affect the absolute value of the determinant - only the sign of the determinant. For example, a square formed by $\mathbf{e}_1$ and $\mathbf{e}_2$ in $\mathbb{R}^2$ is no different from a square formed by $\mathbf{e}_2$ and $\mathbf{e}_1$, except that as we swap the location of the two vectors, the square is turned over on the plane. So we add a sign to the determinant to determine the orientation of the basis.

(c) can be thought of as an initial condition in support of (a). (a) alone is not capable of making $\det$ a unique function, since $\det(I_n)$ can be assigned any values. But by fixing that value to 1, we make the function unique, as we will see in the coming theorems. Also, it is a justified initial condition since the column vectors of I_n form a unit cube above $\mathbb{R}^n$, which has a volume of 1.

With Definition 5.1, we go over some properties of determinants that will be useful in the coming theorems.

Theorem 5.2. *If $\mathbf{a}_i = \mathbf{a}_j$ for some $i \neq j$, then $\det[\cdots \mathbf{a}_i \cdots \mathbf{a}_j \cdots] = 0$.*

Proof. Due to Definition 5.1 (b),

$$\det[\cdots \mathbf{a}_i \cdots \mathbf{a}_j \cdots] = -\det[\cdots \mathbf{a}_j \cdots \mathbf{a}_i \cdots].$$

But since $\mathbf{a}_i = \mathbf{a}_j$, the determinant on both sides becomes 0. ■

We now try showing that $\det$ is a unique function with an explicit formula. To do this, we need to be familiar with a special function called a permutation.

Definition 5.3. *If $n \in \mathbb{N}$, a **permutation** of order n is a bijection between $\{1, 2, \dots, n\}$ and $\{1, 2, \dots, n\}$. S_n denotes the set of all permutations of order n .*

If σ is a permutation in S_n , we can think of the list of values $\sigma(1), \dots, \sigma(n)$ as a list of integers $1, \dots, n$ where the positions of the integers have been swapped several times.

Theorem 5.4. *There exists a unique $\det : M_{n \times n}(\mathbb{R}) \rightarrow \mathbb{R}$ defined as*

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n}, \quad (5.1)$$

where $A = [a_{ij}]$.

Proof. First, check that (5.1) satisfies all three properties in Definition 5.1. This claims the existence of the determinant. Next, we shall claim the uniqueness of the determinant. Let $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n]$ be an $n \times n$ matrix where

$$\begin{aligned} \mathbf{a}_1 &= a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2 + \cdots + a_{n1}\mathbf{e}_n \\ \mathbf{a}_2 &= a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2 + \cdots + a_{n2}\mathbf{e}_n \\ &\vdots \\ \mathbf{a}_n &= a_{1n}\mathbf{e}_1 + a_{2n}\mathbf{e}_2 + \cdots + a_{nn}\mathbf{e}_n \end{aligned} \quad (5.2)$$

Then, the determinant of A can be expressed as

$$\det[\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n] = \sum_{\sigma \in S_n} a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n} \det[\mathbf{e}_{\sigma(1)} \ \mathbf{e}_{\sigma(2)} \ \cdots \ \mathbf{e}_{\sigma(n)}], \quad (5.3)$$

by substituting (5.2) and using Definition 5.1 (a) to expand the terms. Although expanding the terms would lead to n^2 terms, it ends up with only n terms in (5.3) since other terms are erased by Theorem 5.2.

Using Definition 5.1 (b), we can also induce that

$$\det[\mathbf{e}_{\sigma(1)} \ \mathbf{e}_{\sigma(2)} \ \cdots \ \mathbf{e}_{\sigma(n)}] = (-1)^N = \text{sgn}(\sigma)$$

where N is the number of swaps it takes to turn the list $\sigma(1), \dots, \sigma(n)$ back into $1, \dots, n$. $\text{sgn}(\sigma)$ is the **sign** of the permutation σ , defined as $(-1)^N$. Therefore, we have the formula

$$\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n},$$

and it is proven that the determinant is unique. ■

We have found the explicit formula of $\det$ and proven that $\det$ in Definition 5.1 is unique. But it happens that using (5.1) to find the determinant of a square matrix is not easy. Hence, we suggest a different method, called the **cofactor expansion**, to calculate the determinant.

Definition 5.5. Let A be an $n \times n$ matrix, and suppose A_{ij} denotes an $(n-1) \times (n-1)$ matrix obtained by deleting the i th row and j th row of A . Then, the **minor** of a_{ij} is $M_{ij} = \det(A_{ij})$, and the **cofactor** of a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$.

Theorem 5.6. If $A = [a_{ij}]$ is an $n \times n$ matrix and k is an integer between 1 and n , then

$$\det(A) = \sum_{l=1}^n (-1)^{k+l} a_{kl} \det(A_{kl}). \quad (5.4)$$

Proof. Use mathematical induction on n , and for each step, show that the right hand side of (5.4) satisfies Definition 5.1 (a), (b), and (c). Specifics are left as **Exercise**. ■

Example 5.7. Suppose A is a 2×2 matrix such that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Using cofactor expansion on the 1st row, we have

$$\det(A) = (-1)^2 ad + (-1)^3 bc = ad - bc.$$

Example 5.8. Let A be a 3×3 matrix such that

$$A = \begin{bmatrix} 1 & 5 & -1 \\ 3 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix}.$$

Calculate $\det(A)$ using cofactor expansion.

Solution. Suppose we use cofactor expansion on the 2nd row. Then,

$$\begin{aligned} \det(A) &= (-3) \cdot \begin{vmatrix} 5 & -1 \\ 4 & 3 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} + (-3) \cdot \begin{vmatrix} 1 & 5 \\ 1 & 4 \end{vmatrix} \\ &= (-3) \cdot 19 + 2 \cdot 4 + (-3) \cdot (-1) = -46. \end{aligned}$$

5.2 Properties of Determinants

Now we observe the many properties of determinants useful in calculating the determinants of large, complicated matrices.

Basic Properties

Theorem 5.9. Suppose A is an $n \times n$ matrix. Then, $\det(A) = \det(A^T)$.

Proof. By (5.1),

$$\det(A^T) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)},$$

which can be rearranged, using the inverse function σ^{-1} , as

$$\det(A^T) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{\sigma^{-1}(1)1} a_{\sigma^{-1}(2)2} \cdots a_{\sigma^{-1}(n)n}.$$

It should be evident that $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1})$ since $\sigma \circ \sigma^{-1}$ is the identity function.

Therefore,

$$\begin{aligned}\det(A^T) &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma^{-1}) a_{\sigma^{-1}(1)1} a_{\sigma^{-1}(2)2} \cdots a_{\sigma^{-1}(n)n} \\ &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n} = \det(A),\end{aligned}$$

since the sum over all $\sigma^{-1} \in S_n$ is the same as the sum over all $\sigma \in S_n$. ■

Recall that in Theorem 5.6, we examined an efficient method of calculating determinants, called cofactor expansion. We also proved that we could apply cofactor expansion on any row we prefer, as it gives the same result.

Now refer back to Theorem 5.9, where we said that an $n \times n$ matrix A and its transpose A^T have the same determinants. Since applying cofactor expansion on some row of A^T is the same as applying cofactor expansion on some column of A , we can see that it does not matter whether we choose a row or a column to apply cofactor expansion. Hence,

$$\begin{aligned}\det(A) &= \sum_{l=1}^n (-1)^{k+l} a_{kl} \det(A_{kl}) \\ &= \sum_{l=1}^n (-1)^{l+k} a_{lk} \det(A_{lk}),\end{aligned}$$

where k is some integer between 1 and n .

Theorem 5.10. *If A, B are $n \times n$ matrices, then $\det(AB) = \det(A) \cdot \det(B)$.*

Proof. Suppose $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n]$ and $B = [b_{ij}]$. Then, the i th column of AB is

$$\mathbf{c}_i(AB) = A\mathbf{c}_i(B) = b_{1i}\mathbf{a}_1 + b_{2i}\mathbf{a}_2 + \cdots + b_{ni}\mathbf{a}_n.$$

Substituting this into (5.3), we get

$$\begin{aligned}\det(AB) &= \sum_{\sigma \in S_n} b_{\sigma(1)1} b_{\sigma(2)2} \cdots b_{\sigma(n)n} \det[\mathbf{a}_{\sigma(1)} \ \mathbf{a}_{\sigma(2)} \ \cdots \ \mathbf{a}_{\sigma(n)}] \\ &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) b_{\sigma(1)1} b_{\sigma(2)2} \cdots b_{\sigma(n)n} \det[\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n] \\ &= \left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) b_{\sigma(1)1} b_{\sigma(2)2} \cdots b_{\sigma(n)n} \right) \cdot \det(A).\end{aligned}$$

Since the terms inside the parentheses is equal to $\det B$, the theorem is proven. ■

The above theorem presents one of the most important properties of the determinant. It will be often used to prove theorems to come.

Corollary 5.11. *If A is invertible, then $\det(A) \cdot \det(A^{-1}) = 1$.*

Proof. Since $AA^{-1} = A^{-1}A = I_n$,

$$\det(A) \cdot \det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1$$

by Theorem 5.10. ■

So the determinant of an inverse matrix is the reciprocal of the determinant of the given matrix.

Theorem 5.12. *If $A = [a_{ij}]$ is an $n \times n$ triangular matrix, then*

$$\det(A) = a_{11}a_{22} \cdots a_{nn}.$$

Proof. **Exercise.** ■

In other words, the determinant of a triangular matrix is the product of all entries on the main diagonal.

Example 5.13. *Let A be a 3×3 matrix given as*

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & -3 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

Apply cofactor expansion about the third row of A to obtain

$$\det(A) = 2 \cdot \begin{vmatrix} 1 & 3 \\ 0 & -3 \end{vmatrix} = 2 \cdot (-3) = -6.$$

Now check that $\det(A)$ is equal to the product of the entries on the main diagonal.

Theorem 5.14. *For some $n \times n$ matrix A and scalar $c \in \mathbb{R}$, $\det(cA) = c^n \det(A)$,*

Proof. **Exercise.** ■

This theorem disproves a common misconception that $\det(cA) = c \det(A)$. This is true only when A is a 1×1 matrix. In general, it follows the above theorem, so always check the size of A .

Example 5.15. *Prove that if A is a 4×4 matrix, then $\det(A) = \det(-A)$.*

Solution. *Using Theorem 5.14, we have*

$$\det(-A) = (-1)^4 \det(A) = \det(A).$$

Therefore, the determinant of A and $-A$ are equal. Note that this result can be generalized to all square matrices with even numbers of rows and columns.

Determinants and Invertibility

In the context of linear systems, the determinant has to do with the existence and uniqueness of the solution, and in the context of matrices, the determinant has to do with the invertibility of a matrix. We end this chapter by examining these linkages.

First, we look at the determinant of elementary matrices and the effect of elementary row operations on the determinant. Recall that there are three types of elementary row operations:

1. $c \cdot R_i$ (Multiply c on the i th row.) ($c \neq 0$)
2. $R_i \leftrightarrow R_j$ (Exchange between i th and j th row.)
3. $cR_i + R_j$ (Add c multiplied to the i th row on to the j th row.)

Theorem 5.16. *Suppose E is an $n \times n$ elementary matrix. Then:*

- (a) *If E is obtained by applying row operation $c \cdot R_i$ to I_n , then $\det(E) = c$.*
- (b) *If E is obtained by applying row operation $R_i \leftrightarrow R_j$ to I_n , then $\det(E) = -1$.*
- (c) *If E is obtained by applying row operation $cR_i + R_j$ to I_n , then $\det(E) = 1$.*

Proof. (a) Using Definition 5.1 (a), we have $\det(E) = c \det(I_n) = c$.

(b) Applying $R_i \leftrightarrow R_j$ once to I_n is the same as swapping two columns in I_n once. Hence, by Definition 5.1 (b), $\det(E) = -\det(I_n) = -1$.

(c) Notice that E is a triangular matrix. Since the main diagonal of I_n is not changed as it is turned into E , $\det(E) = 1$ by Theorem 5.14. ■

Suppose A is an $m \times n$ matrix. By the above theorem and Theorem 5.10, if A' is a matrix obtained by applying $c \cdot R_i$ to A , then

$$\det(A') = \det(EA) = \det(E) \det(A) = c \cdot \det(A).$$

If A' is obtained by applying $R_i \leftrightarrow R_j$ to A , then

$$\det(A') = \det(EA) = \det(E) \det(A) = -\det(A).$$

And if A' is obtained by applying $cR_i + R_j$ to A , then

$$\det(A') = \det(EA) = \det(E) \det(A) = \det(A).$$

We now add another equivalent statement to Theorem 4.32 using determinants.

Theorem 5.17. *If A is an $n \times n$ matrix, then the following are equivalent:*

- (e) *A is invertible.*
- (r) $\det(A) \neq 0$.

Proof. (e) $\Rightarrow$ (r) Suppose A is invertible. Then, A can be expressed as a product of elementary matrices $E_1, \dots, E_k$, due to Theorem 3.64. Hence,

$$\det(A) = \det(E_1 \cdots E_k) = \det(E_1) \cdots \det(E_k) \neq 0$$

because elementary matrices always have a nonzero determinant.

(r) $\Rightarrow$ (e) Suppose $\det(A) \neq 0$. If R is the reduced row echelon form of A , then there exists elementary matrices $E'_1, \dots, E'_k$ such that $E'_1 \cdots E'_k A = R$. Thus,

$$\det(R) = \det(E'_1) \cdots \det(E'_k) \det(A) \neq 0.$$

This implies that $\det(R)$ cannot have a zero row, giving $R = I_n$. Hence, A is invertible. $\blacksquare$

The intuition for the above theorem is as follows. If A is an $n \times n$ matrix and $\det(A) = 0$, it means that the image of the unit cube in $\mathbb{R}^n$ has a volume of 0. In other words, the dimension of the image has shrunk from the unit cube. Hence, if $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear map defined as $T(\mathbf{x}) = A\mathbf{x}$, then T is not onto $\mathbb{R}^n$ and $A\mathbf{x} = \mathbf{b}$ is not consistent for all $\mathbf{b}$ in $\mathbb{R}^n$. This implies that A is not invertible.

Now we observe another method for finding the inverse of a matrix A . This time, we use determinants and the adjoint of A , which is defined below.

Definition 5.18. Suppose $A = [a_{ij}]$ is an $n \times n$ matrix. If C_{ij} denotes the cofactor of a_{ij} , then

$$\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

is called the **adjoint** of A and denoted as $\text{adj}(A)$.

The following theorem gives an explicit formula for A^{-1} using the determinant and the adjoint matrix. We can use this formula as another way of finding inverse matrices, other than the inversion algorithm we examined in Chapter 3.

Theorem 5.19. If A is invertible, then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A).$$

Proof. Claim that $A \cdot \text{adj}(A) = \det(A)I$. If $A = [a_{ij}]$ and $\text{adj}(A) = [C_{ji}]$, then the ij -th entry of $A \cdot \text{adj}(A)$ is

$$\mathbf{r}_i(A) \cdot \mathbf{c}_j(\text{adj}(A)) = a_{i1}C_{j1} + \cdots + a_{in}C_{jn}.$$

We now divide into two cases. First, if $i = j$, then

$$\mathbf{r}_i(A) \cdot \mathbf{c}_j(\text{adj}(A)) = a_{i1}C_{i1} + \cdots + a_{in}C_{in} = \det(A).$$

since $a_{i1}C_{i1} + \cdots + a_{in}C_{in}$ is the cofactor expansion of A along the i th row.

Second, if $i \neq j$, consider a matrix A' obtained by replacing the j th row of A with the i th row of A . Then, $\det(A') = 0$ since A' has two identical rows, and its cofactor expansion along the j th row gives

$$a_{i1}C_{j1} + \cdots + a_{in}C_{jn} = 0.$$

Hence,

$$\mathbf{r}_i(A) \cdot \mathbf{c}_j(\text{adj}(A)) = a_{i1}C_{j1} + \cdots + a_{in}C_{jn} = 0.$$

Therefore, $A \cdot \text{adj}(A) = \det(A)I$ and

$$A \cdot \left(\frac{1}{\det(A)} \text{adj}(A) \right) = I_n,$$

which proves the theorem. ■

Finally, we look at an important theorem called Cramer's rule. It suggests a method using determinants to solve a given linear system when we are guaranteed its existence and uniqueness of solution.

Theorem 5.20 (Cramer's Rule). *Suppose A is an invertible $n \times n$ matrix such that $A = [\mathbf{a}_1 \cdots \mathbf{a}_n]$ and $\mathbf{b} = [b_{ij}]$ is a column vector of size $n \times 1$. Then, the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution $(x_1, \dots, x_n)$ that satisfies*

$$x_i \det(A) = \det[\mathbf{a}_1 \cdots \mathbf{a}_{i-1} \ \mathbf{b} \ \mathbf{a}_{i+1} \cdots \mathbf{a}_n] \quad \text{for all } i = 1, \dots, n.$$

Proof. If $(x_1, \dots, x_n)$ is a solution of $A\mathbf{x} = \mathbf{b}$, then $\mathbf{b} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$. Using (5.1) and Definition 5.1 (a), we get

$$\det[\mathbf{a}_1 \cdots \mathbf{a}_{i-1} \ \mathbf{b} \ \mathbf{a}_{i+1} \cdots \mathbf{a}_n] = x_i \det[\mathbf{a}_1 \cdots \mathbf{a}_n],$$

and the theorem is proven. ■

Using the above theorem, we can find all x_i by calculating the determinants of the matrices A and $[\mathbf{a}_1 \cdots \mathbf{a}_{i-1} \ \mathbf{b} \ \mathbf{a}_{i+1} \cdots \mathbf{a}_n]$, which is the matrix obtained by replacing the i th column of A with $\mathbf{b}$.

Example 5.21. *Consider the linear system*

$$\begin{cases} x_2 + 2x_3 = 8 \\ 2x_1 - 2x_2 + 4x_3 = 10 \\ 3x_1 - x_2 - 2x_3 = -5 \end{cases}$$

and let

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & -2 & 4 \\ 3 & -1 & -2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 8 & 1 & 2 \\ 10 & -2 & 4 \\ -5 & -1 & -2 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 8 & 2 \\ 2 & 10 & 4 \\ 3 & -5 & -2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 & 8 \\ 2 & -2 & 10 \\ 3 & -1 & -5 \end{bmatrix}.$$

Then by Cramer's rule,

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad x_3 = \frac{\det(A_3)}{\det(A)}.$$

Hence, compute the determinants of A, A_1, A_2, A_3 to obtain the solution of the given linear system. Since

$$\det(A) = 24, \quad \det(A_1) = 24, \quad \det(A_2) = 48, \quad \det(A_3) = 72,$$

the solution of the given linear system is $x_1 = 1, x_2 = 2, x_3 = 3$.

6 | Diagonalization

Eigenvalues and eigenvectors are special types of scalars and vectors with wide range of usage in linear algebra, matrix theory, differential equations, and applied fields like computer graphics. They are an intrinsic property of square matrices, and give us further understanding of how square matrices work.

We also look further into similar matrices we defined back in Definition 4.34. We will check that similar matrices share various characteristics, although they are different matrices.

Finally, we discuss the diagonalization of square matrices, a process which simplifies the given square matrix in the form that calculations are much easier. We look at what it means for a square matrix to be diagonalizable.

6.1 Eigenvalues and Eigenvectors

Consider an $n \times n$ matrix A and a linear map $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Eigenvectors are nonzero vectors in $\mathbb{R}^n$ such that the map T preserves its direction. In other words, if $\mathbf{x}_0$ is an eigenvector, $\mathbf{x}_0$ and $T(\mathbf{x}_0)$ are parallel. The scaling factor between $\mathbf{x}_0$ and $T(\mathbf{x}_0)$ is called an eigenvalue. The following definition summarizes this.

Definition 6.1. Suppose A is an $n \times n$ matrix. If there exists a nonzero vector $\mathbf{x}$ in $\mathbb{R}^n$ such that

$$A\mathbf{x} = \lambda\mathbf{x} \tag{6.1}$$

for some scalar λ , then λ is called the **eigenvalue** of A and $\mathbf{x}$ is called the **eigenvector** of A corresponding to λ .

What is so special about eigenvalues and eigenvectors? Since the direction of eigenvectors are not affected by A , the span of the eigenvectors is not affected by A either. This property is neatly used to form advanced theories of linear algebra. But first, we look at how we find the eigenvalues of a given square matrix.

Theorem 6.2. If A is an $n \times n$ matrix, λ is an eigenvalue of A if and only if

$$\det(A - \lambda I) = 0.$$

Proof. (6.1) can be rewritten as the homogeneous linear system

$$(A - \lambda I)\mathbf{x} = \mathbf{0}.$$

By the equivalence of Theorem 3.64 (b) and Theorem 5.17 (r), the above linear system has a nontrivial solution if and only if $\det(A - \lambda I) = 0$. ■

By the above theorem, we can find the eigenvalue of a given square matrix by solving the equation $\det(A - \lambda I) = 0$, where the equation is called the **characteristic equation** of A and $\det(A - \lambda I)$ is called the **characteristic polynomial** of A . Notice that since the determinant gives a scalar, we are given a single equation to find λ . Also, the determinant can be expressed as a degree n polynomial of λ whose leading coefficient of λ^n is $(-1)^n$. We check this in the following example.

Example 6.3. Consider the 3×3 matrix

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 3 \\ 3 & 2 & -1 \end{bmatrix}.$$

Since

$$A - \lambda I = \begin{bmatrix} 1 - \lambda & 0 & -2 \\ 2 & 1 - \lambda & 3 \\ 3 & 2 & -1 - \lambda \end{bmatrix},$$

the characteristic polynomial of A becomes

$$\det(A - \lambda I) = -\lambda^3 + \lambda^2 - 8.$$

By the fundamental theorem of algebra, a degree n polynomial equation has at most n distinct complex roots. Hence, an $n \times n$ matrix may have at most n distinct eigenvalues in $\mathbb{C}$.

Before moving onto finding eigenvectors, we look at some properties of eigenvalues. The following theorems give easier methods to find eigenvalues for special matrices.

Theorem 6.4. If A is an $n \times n$ triangular matrix, then the diagonal entries of A are the eigenvalues of A .

Proof. By Theorem 5.12, the determinant of $A = [a_{ij}]$ is the product of all diagonal entries. Since $A - \lambda I$ is also a triangular matrix, the characteristic equation of A becomes $\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) = 0$. Hence, the theorem is proven. ■

Theorem 6.5. Suppose A is an $n \times n$ matrix with eigenvalue λ and a corresponding eigenvector $\mathbf{x}$. Then, for some positive integer k , the matrix A^k has eigenvalue λ^k and a corresponding eigenvector $\mathbf{x}$.

Proof. Since λ is an eigenvalue of A and $\mathbf{x}$ is a corresponding eigenvector, we have $A\mathbf{x} = \lambda\mathbf{x}$. Now use mathematical induction on k . If $k = 2$, then

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) = \lambda^2\mathbf{x}.$$

Hence, A^2 has an eigenvalue λ^2 and a corresponding eigenvector $\mathbf{x}$.

Suppose A^k has an eigenvalue λ^k and a corresponding eigenvector $\mathbf{x}$. Then,

$$A^{k+1}\mathbf{x} = A(A^k\mathbf{x}) = A(\lambda^k\mathbf{x}) = \lambda^k(A\mathbf{x}) = \lambda^{k+1}\mathbf{x}$$

and hence, λ^{k+1} is an eigenvalue of A^k and $\mathbf{x}$ is a corresponding eigenvector. Therefore, the theorem is proven. ■

Theorem 6.6. Suppose A is an invertible $n \times n$ matrix with eigenvalue λ and a corresponding eigenvector $\mathbf{x}$. Then, the inverse A^{-1} has eigenvalue λ^{-1} with corresponding eigenvector $\mathbf{x}$.

Proof. Since $A\mathbf{x} = \lambda\mathbf{x}$, we can multiply A^{-1} and λ^{-1} to both sides to get

$$A^{-1}A\mathbf{x} = \lambda^{-1}\lambda\mathbf{x}.$$

Therefore, A^{-1} has an eigenvalue λ^{-1} and its corresponding eigenvector is $\mathbf{x}$. ■

Theorem 6.7. Suppose A is an $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. Then:

(a) $\text{tr}(A) = \lambda_1 + \dots + \lambda_n$.

(b) $\det(A) = \lambda_1 \cdots \lambda_n$.

where $\text{tr}(A)$, the **trace** of A , is the sum of its diagonal entries.

Proof. (a) If $A = [a_{ij}]$, then

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}.$$

Thus, the λ^{n-1} term of $\det(A - \lambda I)$ involves only the product of the diagonal terms of $A - \lambda I$. Also using that

$$\det(A - \lambda I) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda), \quad (6.2)$$

we have $\text{tr}(A) = a_{11} + \dots + a_{nn} = \lambda_1 + \dots + \lambda_n$.

(b) Substitute $\lambda = 0$ into (6.2) to prove the theorem. ■

Note that in the above theorem, the eigenvalues $\lambda_1, \dots, \lambda_n$ are not necessarily distinct. They are just the solution of the characteristic equation.

Also, (a) implies that we can use eigenvalues to determine the invertibility of the given square matrix. The below theorem shows how it works.

Theorem 6.8. *If A is an $n \times n$ matrix, then the following are equivalent:*

- (e) A is invertible.
- (s) $\lambda = 0$ is not an eigenvalue of A .

Proof. Use Theorem 5.17 and 6.7 (b). Specifics are left as **Exercise**. ■

Why is the eigenvalue related with the invertibility of A ? It is quite similar to the intuition we gave for what $\det(A) = 0$ means. If $\lambda = 0$ is an eigenvalue of A , then there exists nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x} = \mathbf{0}$. Hence, null A would be at least a one-dimensional vector space, implying that the image of some region D in $\mathbb{R}^n$ may be reduced in terms of dimensions.

Finding Eigenvectors

We now move to finding the eigenvectors of a square matrix. Recall that we found the eigenvalues of square matrix A by solving the characteristic equation $\det(A - \lambda I) = 0$. Once we find all eigenvalues of A , we can find the eigenvectors for each eigenvalue by solving the linear system $(A - \lambda I)\mathbf{x} = \mathbf{0}$. The following example illustrates this process.

Example 6.9. *Suppose A is a 2×2 matrix*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}.$$

Then, the characteristic equation of A is

$$\det(A - \lambda I) = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0,$$

which gives us eigenvalues $\lambda_1 = 4$ and $\lambda_2 = -1$. Now let us find the eigenvectors corresponding to $\lambda_1 = 4$. The homogeneous linear system $(A - 4I)\mathbf{x} = \mathbf{0}$, which can be expressed as

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

leads us to solutions $(x_1, x_2) = (2t, 3t)$, where t can be any real number. Hence,

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} t$$

are the eigenvectors corresponding to $\lambda_1 = 4$. To find the eigenvectors corresponding to $\lambda_2 = -1$, solve $(A + I)\mathbf{x} = \mathbf{0}$ in a similar manner. It will give solutions $(x_1, x_2) = (-s, s)$ where s is any real number, implying that

$$\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} s$$

are the eigenvectors corresponding to $\lambda_2 = -1$.

6.2 Diagonalization

So far, we have looked at what eigenvalues and eigenvectors are and how to find them. These concepts turn out to provide a solution for the **diagonalization problem**:

- For square matrix A , is there an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix?

In other words, is A similar to a diagonal matrix? To properly address this problem, we begin with looking at similar matrices in depth.

Similar Matrices

Recall that in Section 4.4, we showed that matrices $[T]_{\mathcal{B}'}$ and $[T]_{\mathcal{B}}$ are similar. We only defined what similar matrices are, so we are yet to have a proper understanding on them.

Two matrices that are similar share various properties, called **similarity invariants**, in common, although they are different matrices. We shall look at some of these invariants through the following theorem.

Theorem 6.10. *If A and B are similar, then:*

- (a) $\det(A) = \det(B)$.
- (b) $\text{rank}(A) = \text{rank}(B)$.
- (c) $\det(A - \lambda I) = \det(B - \lambda I)$.

Proof. (a) Since A and B are similar, let P be an invertible matrix such that $B = P^{-1}AP$. Then,

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) = \det(A).$$

(b) First, claim that $\text{null}(PB) = \text{null } B$. If $\mathbf{x}$ is in $\text{null}(PB)$, then $PB\mathbf{x} = \mathbf{0}$ and multiplying P^{-1} to both sides gives $B\mathbf{x} = \mathbf{0}$. Thus, $\mathbf{x}$ is also in $\text{null } B$.

If $\mathbf{x}$ is in $\text{null } B$, then $B\mathbf{x} = \mathbf{0}$ and multiplying P to both sides gives $PB\mathbf{x} = \mathbf{0}$. Thus, $\mathbf{x}$ is also in $\text{null}(PB)$ and therefore, $\text{null}(PB) = \text{null } B$.

Since $\text{nullity}(PB) = \text{nullity}(B)$, we also have $\text{rank}(PB) = \text{rank}(B)$. This implies that

$$\text{rank}(AP) = \text{rank}(P^T A^T) = \text{rank}(A^T) = \text{rank}(A),$$

and $\text{rank}(B) = \text{rank}(PB) = \text{rank}(AP) = \text{rank}(A)$, using that $AP = PB$. Hence, the theorem is proven.

(c) Again, using that $B = P^{-1}AP$, we have

$$\begin{aligned}\det(B - \lambda I) &= \det(P^{-1}AP - \lambda P^{-1}P) \\ &= \det(P^{-1}(A - \lambda I)P) = \det(A - \lambda I).\end{aligned}$$

Hence, the characteristic equations of A and B are equivalent. ■

From the above theorem, we can directly find out that similar matrices have the same determinant, rank, and characteristic polynomial. Taking this further, we can easily show that similar matrices also share invertibility, trace, nullity, and eigenvalues too.

So back in the context of linear maps, that $[T]_{\mathcal{B}'}$ and $[T]_{\mathcal{B}}$ are similar means that they share all these properties. Although the choice of the basis may affect the representing matrix, the similarity invariants do not change, as long as the matrix represents the same linear map. Therefore, we may consider the similarity invariants as an intrinsic property of linear maps.

Diagonalizability

Now back to the diagonalization problem. That A is similar to a diagonal matrix D means that D is a diagonal matrix with the same determinant, rank, and eigenvalues as A . So D would be able to represent A in a simple manner.

We now go over when a square matrix is diagonalizable, what it means for a matrix to be diagonalizable, and how we diagonalize a matrix.

Definition 6.11. An $n \times n$ matrix A is called **diagonalizable** if there exists an invertible matrix P such that

$$P^{-1}AP = D$$

where D is a diagonal matrix.

Theorem 6.12. An $n \times n$ matrix A is diagonalizable if and only if we can construct a basis for $\mathbb{R}^n$ out of the eigenvectors of A .

Proof. ($\Rightarrow$) Suppose $P = [\mathbf{p}_1 \ \cdots \ \mathbf{p}_n]$ is an invertible matrix such that $AP = PD$, where $D = [d_{ij}]$ is a diagonal matrix. Then,

$$AP = [A\mathbf{p}_1 \ \cdots \ A\mathbf{p}_n] = [d_{11}\mathbf{p}_1 \ \cdots \ d_{nn}\mathbf{p}_n] = PD,$$

which leads to

$$A\mathbf{p}_i = d_{ii}\mathbf{p}_i \quad \text{for all } i = 1, \dots, n. \tag{6.3}$$

Notice that (6.1) and (6.3) are alike and $\mathbf{p}_i$ are nonzero vectors, which implies that d_{ii} is an eigenvalue of A and $\mathbf{p}_i$ is its corresponding eigenvector. Since P is an invertible matrix, the column vectors of P , or the eigenvectors of A , form a basis for $\mathbb{R}^n$. Hence, we can construct a basis for $\mathbb{R}^n$ using the eigenvectors of A .

($\Leftarrow$) Suppose $\mathbf{p}_1, \dots, \mathbf{p}_n$ are eigenvectors of A that correspond to the eigenvalues $\lambda_1, \dots, \lambda_n$ of A and form a basis for $\mathbb{R}^n$. Construct a matrix $P = [\mathbf{p}_1 \ \cdots \ \mathbf{p}_n]$ that is invertible and hence,

$$AP = [A\mathbf{p}_1 \ \cdots \ A\mathbf{p}_n] = [\lambda_1\mathbf{p}_1 \ \cdots \ \lambda_n\mathbf{p}_n] = PD,$$

where

$$(D)_{ij} = \begin{cases} \lambda_i & \text{if } i = j \\ 0 & \text{else.} \end{cases}$$

Therefore, invertible matrix P makes $P^{-1}AP$ a diagonal matrix, giving that A is diagonalizable. ■

From the above proof, we have found an equivalent statement for diagonalizability. For square matrix A to be diagonalizable, we have to find n linearly independent eigenvectors of A that will form a basis for $\mathbb{R}^n$. Also, notice that the diagonal matrix D is consisted of the eigenvalues of A . Of course, these eigenvalues would also be the eigenvalues of D .

Now we shall dive deeper into investigating diagonalizability. First, we give a special name to the set of eigenvectors corresponding to a single eigenvalue.

Definition 6.13. Suppose A is an $n \times n$ matrix with an eigenvalue λ . Then, the *eigenspace* of A corresponding to λ , denoted as E_λ , is defined as

$$E_\lambda = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \lambda\mathbf{x}\}.$$

Notice that $E_\lambda = \text{null}(A - \lambda I)$ and hence, E_λ is a subspace of $\mathbb{R}^n$. Also, according to the value of λ , the eigenspace will include different elements, since it is a null space. So different eigenspaces would only have the zero vector $\mathbf{0}$ in common.

This leads us to wonder how different eigenspaces are related to each other. The following theorem takes a first step into this question.

Theorem 6.14. Let A be an $n \times n$ matrix with k distinct eigenvalues $\lambda_1, \dots, \lambda_k$ and eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ corresponding to each eigenvalue respectively. Then, $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent.

Proof. Suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly dependent, and let m be the maximal index such that $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is linearly independent ($m < k$). Then, there exists scalars $c_1, \dots, c_m$ not all zero such that

$$\mathbf{v}_{m+1} = c_1\mathbf{v}_1 + \dots + c_m\mathbf{v}_m. \quad (6.4)$$

Hence, multiplying A to both sides of (6.4) gives

$$\lambda_{m+1}\mathbf{v}_{m+1} = c_1\lambda_1\mathbf{v}_1 + \dots + c_m\lambda_m\mathbf{v}_m \quad (6.5)$$

and multiplying λ_{m+1} to both sides of (6.4) gives

$$\lambda_{m+1}\mathbf{v}_{m+1} = c_1\lambda_{m+1}\mathbf{v}_1 + \dots + c_m\lambda_{m+1}\mathbf{v}_m. \quad (6.6)$$

Subtracting (6.5) with (6.6) leads to

$$\mathbf{0} = c_1(\lambda_1 - \lambda_{m+1})\mathbf{v}_1 + \dots + c_m(\lambda_m - \lambda_{m+1})\mathbf{v}_m,$$

which is a contradiction since $\lambda_1 - \lambda_{m+1}, \dots, \lambda_m - \lambda_{m+1}$ are all nonzero and $\mathbf{v}_1, \dots, \mathbf{v}_m$ are nonzero vectors. ■

Corollary 6.15. *If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.*

Proof. Due to Theorem 6.14, we may choose n eigenvectors of A , each corresponding to different eigenvalues, such that the eigenvectors form a basis for $\mathbb{R}^n$. Hence, by Theorem 6.12, A is diagonalizable. ■

Theorem 6.14 suggests that choosing an eigenvector from each eigenspace leads to a linearly independent set of vectors. It leads us to the corollary above, which gives that $n \times n$ matrices with n distinct eigenvalues are always diagonalizable. Below is an example of this case.

Example 6.16. *Let A be a 2×2 matrix*

$$A = \begin{bmatrix} -8 & 10 \\ -5 & 7 \end{bmatrix}.$$

First, find the eigenvalues of A by solving $\det(A - \lambda I) = 0$. Since the characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} -8 - \lambda & 10 \\ -5 & 7 - \lambda \end{vmatrix} = \lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3),$$

the eigenvalues of A are 2 and -3 . By Corollary 6.15, A is diagonalizable.

Next, find the eigenvectors corresponding to eigenvalues 2 and -3 . It can be shown that

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

are eigenvectors corresponding to $\lambda = 2$ and $\lambda = -3$, respectively. Hence, construct a 2×2 matrix

$$P = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix},$$

and then we have

$$D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -8 & 10 \\ -5 & 7 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = P^{-1}AP.$$

Therefore, A is similar to the diagonal matrix D . Also, check that the main diagonal entries of D are the eigenvalues of A .

Now what if the matrix does not have n distinct eigenvalues? We shall first prove the general version of Theorem 6.14.

Theorem 6.17. *Let A be an $n \times n$ matrix with k distinct eigenvalues $\lambda_1, \dots, \lambda_k$. If $S_1, \dots, S_k$ are linearly independent subsets of $E_{\lambda_1}, \dots, E_{\lambda_k}$, respectively, then $S_1 \cup \dots \cup S_k$ is linearly independent.*

Proof. Let $S_i = \{\mathbf{v}_{i1}, \dots, \mathbf{v}_{i|S_i|}\}$ for all $i = 1, \dots, n$, and suppose there are scalars $a_{11}, \dots, a_{k|S_k|}$ such that

$$\sum_{i=1}^k \left(\sum_{j=1}^{|S_i|} a_{ij} \mathbf{v}_{ij} \right) = \mathbf{0}. \quad (6.7)$$

Since the terms within the parentheses of (6.7) is a vector in E_{λ_i} ,

$$\sum_{j=1}^{|S_i|} a_{ij} \mathbf{v}_{ij} = \mathbf{0} \quad \text{for all } i = 1, \dots, n,$$

by Theorem 6.14. Also, since S_i is linearly independent for all i , all scalars $a_{11}, \dots, a_{k|S_k|}$ are equal to 0. Hence, $S_1 \cup \dots \cup S_k$ is linearly independent. ■

So regardless of how many linearly independent vectors we extract from each eigenspace, the union of all those vectors is still linearly independent. This result is important because it implies that if $E_{\lambda_1}, \dots, E_{\lambda_k}$ have bases $\mathcal{B}_1, \dots, \mathcal{B}_k$, then $\mathcal{B} = \bigcup_{i=1}^k \mathcal{B}_i$ is a basis for $E_{\lambda_1} + \dots + E_{\lambda_k}$, since we know that $\mathcal{B}$ is linearly independent.

Also, recall that $E_{\lambda_i} \cap E_{\lambda_j} = \{\mathbf{0}\}$ for all i and j , since eigenspaces are null spaces. Hence, $E_{\lambda_1} + \dots + E_{\lambda_k}$ is a direct sum and

$$\dim(E_{\lambda_1} \oplus \dots \oplus E_{\lambda_k}) = \dim(E_{\lambda_1}) + \dots + \dim(E_{\lambda_k}).$$

This gives that if $\dim(E_{\lambda_1} \oplus \dots \oplus E_{\lambda_k}) = m$, then we can not choose more than m eigenvectors of A such that they are linearly independent.

We look at some final few theorems to build upon this.

Definition 6.18. Let A be an $n \times n$ matrix with k distinct eigenvalues $\lambda_1, \dots, \lambda_k$. The **algebraic multiplicity** of eigenvalue λ_i , denoted as m_i , is the largest integer such that $(\lambda_i - \lambda)^{m_i}$ is a factor of $\det(A - \lambda I)$. The **geometric multiplicity** of eigenvalue λ_i , denoted as μ_i , is the dimension of E_{λ_i} .

Theorem 6.19. If A is an $n \times n$ matrix with k distinct eigenvalues $\lambda_1, \dots, \lambda_k$, then:

- (a) $m_1 + \dots + m_k = n$.
- (b) $\mu_i \leq m_i$ for all integer i between 1 and k .

Proof. (a) Since the characteristic polynomial of A is a degree n polynomial and

$$\det(A - \lambda I) = (\lambda_1 - \lambda)^{m_1} \dots (\lambda_k - \lambda)^{m_k},$$

$$m_1 + \dots + m_k = n.$$

(b) Suppose $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is a basis for E_{λ_i} ($m \leq n$). Since E_{λ_i} is a subspace of $\mathbb{R}^n$, we can extend $\mathcal{B}$ such that $\mathcal{B}' = \mathcal{B} \cup \{\mathbf{v}_{m+1}, \dots, \mathbf{v}_n\}$. Hence, we can construct an $n \times n$ invertible matrix $P = [\mathbf{v}_1 \dots \mathbf{v}_n]$ such that

$$AP = [A\mathbf{v}_1 \dots A\mathbf{v}_n] = [\lambda_i \mathbf{v}_1 \dots \lambda_i \mathbf{v}_m \ A\mathbf{v}_{m+1} \dots A\mathbf{v}_n] = PB,$$

where

$$B = \begin{bmatrix} \lambda_i I_m & B' \\ \mathbf{0}_{(n-m) \times m} & B'' \end{bmatrix}.$$

Notice that A and B are similar and share the characteristic equation. Thus,

$$\det(A - \lambda I) = \det(B - \lambda I) = (\lambda_i - \lambda)^m \det(B'' - \lambda I_{n-m}),$$

which gives that m_i is at least m . Therefore, $\mu_i = m \leq m_i$. ■

So to summarize, the sum of all algebraic multiplicities is always equal to n , and the geometric multiplicity is always equal to or smaller than the algebraic multiplicity.

Using this result, we have that if $\dim(E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}) = m$, then $m \leq n$. Based on Theorem 6.12, if $m < n$, we can not form a basis for $\mathbb{R}^n$ out of eigenvectors of A , so A is not diagonalizable.

If $m = n$, then we can always form a basis for $\mathbb{R}^n$ using eigenvectors of A , so A is diagonalizable. Also, $m = n$ is the same as saying that $E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k} = \mathbb{R}^n$, since eigenspaces are subspaces of $\mathbb{R}^n$. Therefore, we obtain the following theorem.

Theorem 6.20. *If A is an $n \times n$ matrix with k distinct eigenvalues $\lambda_1, \dots, \lambda_k$, then the following are equivalent:*

- (a) A is diagonalizable.
- (b) $E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k} = \mathbb{R}^n$.
- (c) $\dim(E_{\lambda_i}) = m_i$ for all integer i between 1 and k .

The equivalence between (b) and (c) can be proven by using Theorem 6.19. Hence, we can now state that an $n \times n$ square matrix is diagonalizable when its eigenvectors form a basis for $\mathbb{R}^n$. We end the discussion of diagonalizability with the following example.

Example 6.21. *Suppose A is a 3×3 matrix given as*

$$A = \begin{bmatrix} -16 & -15 & 15 \\ 15 & 14 & -15 \\ -3 & -3 & 2 \end{bmatrix}.$$

The eigenvalues of A are -1 and 2 , with an algebraic multiplicity of 2 and 1 , respectively. The eigenspace corresponding to -1 is spanned by the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 6 \\ -5 \\ 1 \end{bmatrix},$$

which are linearly independent. Hence, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for E_{-1} , and the geometric multiplicity of $\lambda = -1$ is 2 . The eigenspace corresponding to 2 is spanned

by the vector

$$\mathbf{v}_3 = \begin{bmatrix} 5 \\ -5 \\ 1 \end{bmatrix},$$

which forms a basis for E_2 . Hence, the geometric multiplicity of $\lambda = 2$ is 1, and by Theorem 6.20, A is diagonalizable.

Using $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, construct a 3×3 matrix

$$P = \begin{bmatrix} 1 & 6 & 5 \\ -1 & -5 & -5 \\ 0 & 1 & 1 \end{bmatrix},$$

and we have

$$\begin{aligned} D &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 6 & 5 \\ -1 & -5 & -5 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -16 & -15 & 15 \\ 15 & 14 & -15 \\ -3 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 6 & 5 \\ -1 & -5 & -5 \\ 0 & 1 & 1 \end{bmatrix} = P^{-1}AP. \end{aligned}$$

Therefore, A is similar to diagonal matrix D .

Linear Maps and Diagonalization

We now extend from the diagonalizability of square matrices. What does it mean for a linear map to be diagonalizable?

Consider a linear operator $T : V \rightarrow V$ where V is a vector space with dimension n . That T is a diagonalizable map means that there exists some basis $\mathcal{B}$ for V such that $[T]_{\mathcal{B}}$ is diagonalizable.

Also, we may define the eigenvalue, eigenvector, and eigenspace of T as follows.

Definition 6.22. Let $T : V \rightarrow V$ be a linear operator above vector space V . If there exists a nonzero vector $\mathbf{x}$ in V such that

$$T(\mathbf{x}) = \lambda \mathbf{x}$$

for some scalar λ , then λ is called the **eigenvalue** of T and $\mathbf{x}$ is called the **eigenvector** of T corresponding to λ .

Definition 6.23. Let $T : V \rightarrow V$ be a linear operator above vector space V with an eigenvalue λ . Then, the **eigenspace** of T corresponding to λ , denoted as E_{λ} , is defined as

$$E_{\lambda} = \{\mathbf{x} \in V : T(\mathbf{x}) = \lambda \mathbf{x}\}.$$

Hence, we can rephrase Theorem 6.20 as follows.

Theorem 6.24. *If $T : V \rightarrow V$ is a linear operator above vector space V with k distinct eigenvalues $\lambda_1, \dots, \lambda_k$, then the following are equivalent:*

- (a) *T is diagonalizable.*
- (b) $E_{\lambda_1} \oplus \dots \oplus E_{\lambda_k} = V$.

Powers of Matrices

Suppose A is a diagonalizable $n \times n$ matrix. Hence, let P be an invertible matrix such that $D = P^{-1}AP$ where D is a diagonal matrix. Then,

$$D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A^2P$$

and in general,

$$D^k = P^{-1}A^kP \quad \text{for all } k \in \mathbb{Z}^+. \quad (6.8)$$

Notice that D^k is easy to compute since if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A , then

$$(D^k)_{ij} = \begin{cases} \lambda_i^k & \text{if } i = j \\ 0 & \text{else.} \end{cases}$$

Also, from (6.8), we have that

$$A^k = PD^kP^{-1},$$

which allows us to calculate A^k very easily.

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